WEYL'S ASYMPTOTIC LAW FOR THE DIRICHLET LAPLACIAN: EXEGESIS OF A FRANK-GEISINGER'S THEOREM

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Contents

1. Motivation

1.1. **On the main theorem.** The aim of these notes is to estimate sums of eigenvalues of the Dirichlet Laplacian $-\Delta_\Omega$ where Ω is an open, bounded, and $\mathscr{C}^{1,\alpha}$ subset of $\mathbb{R}^d$. We recall that $-\Delta_\Omega$ is the self-adjoint operator associated with the quadratic form

$$\mathscr{Q}_\Omega(\psi) = \int_\Omega |\nabla\psi|^2 \mathrm{d}x\,, \quad \forall \psi \in H^1_0(\Omega)\,.$$

Let $h > 0$ and consider

$$H_\Omega = -h^2\Delta_\Omega - 1\,.$$

Let us denote by $(\lambda_j)_{j\geqslant 1}$ the non-decreasing sequence of the eigenvalues of $-\Delta_\Omega$. We would like to describe the behavior of the "total energy of the system" below 1,

$$\mathrm{Tr}(H_\Omega)_- = \sum_{j\geqslant 1}(h^2\lambda_j - 1)_-\,,$$

in the semiclassical limit $h \to 0$. Our aim is to explain the proof of the following theorem obtained by Frank and Geisinger. These notes are based on prerequisites in [4, Chapter 6] and are an exegesis of the short paper [3]. This "exegesis" of [3] has been influenced by many interactions with S. Fournais, R. Frank, S. Larson, and T. Østergaard-Sørensen and also by the Ph.D. dissertation of S. Gottwald. The reader can consult the detailed lecture notes [1] about spectral theory.

Theorem 1.1 (Frank&Geisinger '11)**.**

$$\mathrm{Tr}(H_\Omega)_- = L_d|\Omega|h^{-d} - \frac{1}{4}L_{d-1}|\partial\Omega|h^{-d+1} + o(h^{-d+1})\,,$$

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where

$$L_d = (2\pi)^{-d}\int_{\mathbb{R}^d}(\xi^2-1)_-\mathrm{d}\xi = (2\pi)^{-d}\frac{2\omega_d}{d+2}\,.$$

1.2. **Historical context and naive considerations.** Theorem 1.1 is part of a rather long story. This story started with Weyl in [7] and the asympopttic expansion of the counting function

$$N_\Omega(h) = |\Omega|\frac{C_d}{h^d} + o(h^{-d})\,,\quad C_d = \frac{\omega_d}{(2\pi)^d}\,.$$

Here, $N_\Omega(h) = |\{k \geqslant 1 : h^2\lambda_k < 1\}|$. Under the geometric assumption that Ω has no "periodic point" and when Ω is smooth, V. Ivrii proved the second term asymptotics in [5] (see also its translation):

$$N_\Omega(h) = |\Omega|C_d h^{-d} - \frac{1}{4}|\partial\Omega|C_{d-1}h^{-d+1} + o(h^{-d+1})\,. \tag{1.1}$$

In general, Weyl's asymptotic expansions can be obtained by means of microlocal technics. The reader can consult [2] where it is proved, for instance, that

$$|\{\lambda_k(h) < 1\}| = (2\pi h)^{-d}\int_{a(x,\xi)<1}\mathrm{d}x\,\mathrm{d}\xi + o(h^{-d+1})\,,$$

where the $(\lambda_k(h))_{k\geqslant 1}$ are the eigenvalues of an elliptic pseudo-differential operator defined by

$$\mathrm{Op}_h^{\mathrm{W}}(a)\psi(x) = (2\pi h)^{-d}\int_{\mathbb{R}^{2d}} e^{i\langle x-y,\eta\rangle/h} a\left(\frac{x+y}{2},\eta\right)\psi(y)\mathrm{d}y\mathrm{d}\eta\,.$$

A very good introduction to semiclassical/microlocal analysis is the book by Zworski [8].

Sometimes (especially in old references), the Weyl asymptotics is written in terms of a large parameter $\lambda = h^{-\frac{1}{2}}$. The expansion (1.1) can be rewritten as

$$|\{k\geqslant 1 : \lambda_k < \lambda\}| =: N(\lambda) = |\Omega|C_d\lambda^{\frac{d}{2}} - \frac{1}{4}C_{d-1}|\partial\Omega|\lambda^{\frac{d-1}{2}} + o(\lambda^{\frac{d-1}{2}})\,.$$

Note that,

$$\int_0^\lambda N(u)\mathrm{d}u = |\Omega|\frac{2C_d}{d+2}\lambda^{\frac{d}{2}+1} - \frac{1}{4}\frac{2C_{d-1}}{d+1}|\partial\Omega|\lambda^{\frac{d+1}{2}} + o(\lambda^{\frac{d+1}{2}})\,,$$

and also, by definition, of the counting function

$$\int_0^\lambda N(u)\mathrm{d}u = -T(\lambda) + \lambda N(\lambda) = \sum_{k=1}^{N(\lambda)}(\lambda_k-\lambda)_-\,,\quad T(\lambda) = \sum_{k=1}^{N(\lambda)}\lambda_k\,.$$

Coming back to h, we deduce Theorem 1.1, under Ivrii's assumptions. Our aim is to obtain Theorem 1.1 by direct means and under weaker regularity assumptions. For that purpose, one will use basic semiclassical tools, as the ones developped in [6].

Many generalizations of Theorem 1.1 can be considered (in various directions), with the tools introduced in these lecture notes (presence of magnetic fields, Lipschitz boundaries), and are nowadays the subject of active research due to their connexions with quantum chemistry.

2. Local traces

2.1. **Local traces inside.** Let us consider here $\mathscr{L}_h^{\mathbb{R}^d} = -h^2\Delta - 1$ acting on $L^2(\mathbb{R}^d)$.

2.1.1. *Computing kernels.*

Definition 2.1. We define the unitary Fourier transform by

$$\mathscr{F}\psi(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-ix\xi}\psi(x)\mathrm{d}x\,.$$

Lemma 2.2. *In the sense of quadratic forms, we have*

$$(\mathscr{L}_h^{\mathbb{R}^d})_- \leqslant \gamma_h\,, \qquad \gamma_h = \mathbb{1}_{\mathbb{R}^-}(\mathscr{L}_h^{\mathbb{R}^d})\,.$$

Consider $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^2)$. Then,

$$\varphi(\mathscr{L}_h^{\mathbb{R}^d})_-\varphi \leqslant \varphi\gamma_h\varphi\,.$$

Lemma 2.3. *We have*

$$\mathscr{F}\mathscr{L}_h\mathscr{F}^{-1} = h^2\xi^2 - 1\,.$$

In particular,

$$\gamma_h = \mathscr{F}^{-1}\mathbb{1}_{\mathbb{R}_-}(h^2\xi^2-1)\mathscr{F}\,,$$

and

$$\begin{aligned}\gamma_h\psi(x) &= (2\pi)^{-d}\int_{\mathbb{R}^{2d}} e^{i(x-y)\xi}\mathbb{1}_{\mathbb{R}_-}(h^2\xi^2-1)\psi(y)\mathrm{d}y\mathrm{d}\xi\\ &= (2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d}\mathrm{d}y\psi(y)\mathscr{F}^{-1}(\mathbb{1}_{\mathbb{R}_-}(h^2\xi^2-1))(x-y)\,.\end{aligned}$$

Proposition 2.4. *Consider $\varphi \in \mathscr{C}_0^0(\mathbb{R}^2)$. Then,*

(i) *The bounded operator $\varphi\gamma_h$ is Hilbert-Schmidt and*

$$\|\varphi\gamma_h\|_2^2 = \frac{\omega_d}{(2\pi h)^d}\|\varphi\|^2\,.$$

(ii) *The bounded self-adjoint operator $\varphi\gamma_h\varphi$ is trace-class. Moreover,*

$$\mathrm{Tr}(\varphi\gamma_h\varphi) = \frac{\omega_d}{(2\pi h)^d}\|\varphi\|^2\,. \tag{2.1}$$

Proof. For the first item, we notice that the kernel K of $\varphi\gamma_h$ is given by

$$K(x,y) = (2\pi)^{-\frac{d}{2}}\varphi(x)\mathscr{F}^{-1}(\mathbb{1}_{\mathbb{R}_-}(h^2\xi^2-1))(x-y)\,.$$

From the Parseval formula, we see that $K \in L^2(\mathbb{R}^{2d})$ and

$$\|K\|^2_{L^2(\mathbb{R}^{2d})} = (2\pi)^{-d}\|\varphi\|^2\int_{\mathbb{R}^d}\mathrm{d}\xi\mathbb{1}_{\mathbb{R}_-}(h^2\xi^2-1)\,.$$

For the second item, it is sufficient to notice that

$$\varphi\gamma_h\varphi = \varphi\gamma_h\gamma_h\varphi = (\varphi\gamma_h)(\varphi\gamma_h)^*\,,$$

and

$$\mathrm{Tr}(\varphi\gamma_h\varphi) = \|\varphi\gamma_h\|_2^2\,.$$

$\square$

Corollary 2.5. *For all $\varphi \in \mathscr{C}_0^1(\mathbb{R}^2)$, $\varphi(\mathscr{L}_h^{\mathbb{R}^d})_-\varphi$ is trace-class. Moreover, $(\varphi\mathscr{L}_h^{\mathbb{R}^d}\varphi)_-$ is also trace-class and*

$$\mathrm{Tr}(\varphi\mathscr{L}_h\varphi)_- \leqslant \mathrm{Tr}(\varphi(\mathscr{L}_h)_-\varphi)\,.$$

Moreover,

$$\mathrm{Tr}(\varphi(\mathscr{L}_h^{\mathbb{R}^d})_-\varphi) = (2\pi)^{-d}h^{-d}\|\varphi\|^2\int_{\mathbb{R}^d}(\xi^2-1)_-\mathrm{d}\xi\,.$$

Proof. The first part of the statement follows from Lemma 2.2 and Proposition 2.4. For the second part, we consider a Hilbert basis $(\psi_j)_{j\geqslant 1}$ such that $(\psi_j)_{j\in J}$ is a Hilbert basis of the negative (Hilbert) subspace of $\varphi\mathscr{L}_h^{\mathbb{R}^d}\varphi$. Then, for all $j\in J$,

$$\langle\psi_j,(\varphi\mathscr{L}_h^{\mathbb{R}^d}\varphi)_-\psi_j\rangle=-\langle\psi_j,(\varphi\mathscr{L}_h^{\mathbb{R}^d}\varphi)\psi_j\rangle\leqslant\langle\psi_j,(\varphi(\mathscr{L}_h^{\mathbb{R}^d})_-\varphi)\psi_j\rangle\,,$$

and, for all $j\in\mathbb{N}^*\setminus J$, $\langle\psi_j,(\varphi\mathscr{L}_h^{\mathbb{R}^d}\varphi)_-\psi_j\rangle=0$. This shows that $\sum_{j\geqslant 1}\langle\psi_j,(\varphi\mathscr{L}_h^{\mathbb{R}^d}\varphi)_-\psi_j\rangle$ is convergent, that the non-negative operator $(\varphi\mathscr{L}_h^{\mathbb{R}^d}\varphi)_-$ is trace-class, and the inequality follows. Then, we write

$$\varphi(\mathscr{L}_h^{\mathbb{R}^d})_-\varphi=\varphi(\mathscr{L}_h^{\mathbb{R}^d})_-^{\frac12}\left(\varphi(\mathscr{L}_h^{\mathbb{R}^d})_-^{\frac12}\right)^*\,.$$

The kernel of $\varphi(\mathscr{L}_h^{\mathbb{R}^d})_-^{\frac12}$ is

$$(2\pi)^{-\frac{d}{2}}\varphi(x)\mathscr{F}^{-1}((h^2\xi^2-1)_-^{\frac12})(x-y)\,.$$

Therefore, it is Hilbert-Schmidt and

$$\|\varphi(\mathscr{L}_h^{\mathbb{R}^d})_-^{\frac12}\|_2^2=(2\pi)^{-d}h^{-d}\|\varphi\|^2\int_{\mathbb{R}^d}(\xi^2-1)_-\mathrm{d}\xi\,.$$

□

2.1.2. *When the minus goes inside.*

Proposition 2.6. *There exist $C,h_0>0$ such that, for all $h\in(0,h_0)$, and all $\varphi\in\mathscr{C}_0^1(\mathbb{R}^d)$,*

$$|\mathrm{Tr}(\varphi\mathscr{L}_h^{\mathbb{R}^d}\varphi)_--\mathrm{Tr}(\varphi(\mathscr{L}_h^{\mathbb{R}^d})_-\varphi)|\leqslant Ch^{2-d}\|\nabla\varphi\|^2\,.$$

Proof. Consider $\chi\in\mathscr{C}_0^\infty(\mathbb{R}^2)$ such that $\varphi\chi=\varphi$. Since $\chi\gamma_h\chi$ is trace-class and $0\leqslant\chi\gamma_h\chi\leqslant 1$, the Variational Principle provides us with

$$-\mathrm{Tr}(\varphi\mathscr{L}_h^{\mathbb{R}^d}\varphi)_-\leqslant\mathrm{Tr}((\varphi\mathscr{L}_h^{\mathbb{R}^d}\varphi)(\chi\gamma_h\chi))\,. \tag{2.2}$$

We write

$$\chi\gamma_h\chi=\sum_{j\geqslant 1}\mu_j\,|\psi_j\rangle\,\langle\psi_j|\ ,$$

so that

$$\chi\gamma_h\chi\psi_j=\mu_j\psi_j\,.$$

Note that, if $\varphi\in\mathscr{C}_0^\infty(\mathbb{R}^d)$,

$$\langle\psi_j,(\varphi\mathscr{L}_h^{\mathbb{R}^d}\varphi)\psi_j\rangle=\langle\psi_j,(\mathscr{L}_h^{\mathbb{R}^d}\varphi^2+[\varphi,\mathscr{L}_h^{\mathbb{R}^d}]\varphi)\psi_j\rangle=\langle\psi_j,(\varphi^2\mathscr{L}_h^{\mathbb{R}^d}+\varphi[\mathscr{L}_h^{\mathbb{R}^d},\varphi])\psi_j\rangle\,,$$

so that

$$2\langle\psi_j,(\varphi\mathscr{L}_h^{\mathbb{R}^d}\varphi)\psi_j\rangle=\langle\psi_j,\left(\varphi^2\mathscr{L}_h^{\mathbb{R}^d}+\mathscr{L}_h\varphi^2-[\varphi,[\varphi,\mathscr{L}_h^{\mathbb{R}^d}]]\right)\psi_j\rangle\,,$$

and thus

$$\langle\psi_j,(\varphi\mathscr{L}_h^{\mathbb{R}^d}\varphi)\psi_j\rangle=\mathrm{Re}\,\langle\psi_j,\left(\varphi^2\mathscr{L}_h^{\mathbb{R}^d}-\frac12[\varphi,[\varphi,\mathscr{L}_h^{\mathbb{R}^d}]]\right)\psi_j\rangle\,,$$

since

$$[\varphi,[\varphi,\mathscr{L}_h^{\mathbb{R}^d}]]=-2h^2|\nabla\varphi|^2\,.$$

Thus, if $\varphi\in\mathscr{C}_0^\infty(\mathbb{R}^d)$,

$$\mathscr{Q}_h(\varphi\psi_j)=\langle\psi_j,(\varphi\mathscr{L}_h^{\mathbb{R}^d}\varphi)\psi_j\rangle=\mathrm{Re}\,\langle\psi_j,\varphi^2\mathscr{L}_h^{\mathbb{R}^d}\psi_j\rangle+h^2\langle\psi_j,|\nabla\varphi|^2\psi_j\rangle\,.$$

This formula can be extended to $\varphi\in\mathscr{C}_0^1(\mathbb{R}^d)$. We have

$$\mu_j\langle\psi_j,(\varphi\mathscr{L}_h^{\mathbb{R}^d}\varphi)\psi_j\rangle=\mathrm{Re}\,\langle\psi_j,\varphi^2\mathscr{L}_h^{\mathbb{R}^d}(\chi\gamma_h\chi)\psi_j\rangle+h^2\langle\psi_j,|\nabla\varphi|^2(\chi\gamma_h\chi)\psi_j\rangle\,.$$

Since $\mathscr{L}_h^{\mathbb{R}^d}$ is a local operator, we have

$$\mu_j\langle\psi_j,(\varphi\mathscr{L}_h^{\mathbb{R}^d}\varphi)\psi_j\rangle=-\mathrm{Re}\,\langle\psi_j,\varphi^2\chi(\mathscr{L}_h^{\mathbb{R}^d})_-\chi)\psi_j\rangle+h^2\langle\psi_j,|\nabla\varphi|^2\chi\gamma_h\chi\psi_j\rangle\,.$$

Now, we observe that $\varphi^2\chi(\mathscr{L}_h^{\mathbb{R}^d})_-\chi)$ and $|\nabla\varphi|^2\chi\gamma_h\chi$ are trace-class. In particular, the series $\sum_{j\geqslant 1}\mu_j\langle\psi_j,(\varphi\mathscr{L}_h^{\mathbb{R}^d}\varphi)\psi_j\rangle$ is convergent and

$$\sum_{j\geqslant 1}\mu_j\langle\psi_j,(\varphi\mathscr{L}_h^{\mathbb{R}^d}\varphi)\psi_j\rangle = -\mathrm{Tr}(\varphi^2\chi(\mathscr{L}_h^{\mathbb{R}^d})_-\chi) + h^2\mathrm{Tr}(|\nabla\varphi|^2\chi\gamma_h\chi)\,.$$

By using the cyclicity of the trace, we also get

$$\sum_{j\geqslant 1}\mu_j\langle\psi_j,(\varphi\mathscr{L}_h^{\mathbb{R}^d}\varphi)\psi_j\rangle = -\mathrm{Tr}(\varphi(\mathscr{L}_h^{\mathbb{R}^d})_-\varphi) + h^2\mathrm{Tr}(|\nabla\varphi|\gamma_h|\nabla\varphi|)\,.$$

This shows that the left-hand side does not depend on the choice of the diagonalizing Hilbert basis $(\psi_j)_{j\geqslant 1}$. By definition, the left-hand side is $\mathrm{Tr}(\varphi\mathscr{L}_h^{\mathbb{R}^d}\varphi(\chi\gamma_h\chi))$, and we have proved that

$$\mathrm{Tr}(\varphi\mathscr{L}_h^{\mathbb{R}^d}\varphi(\chi\gamma_h\chi)) = -\mathrm{Tr}(\varphi(\mathscr{L}_h^{\mathbb{R}^d})_-\varphi) + h^2\mathrm{Tr}(|\nabla\varphi|\gamma_h|\nabla\varphi|)\,.$$

By (2.2), this implies that

$$\mathrm{Tr}(\varphi\mathscr{L}_h^{\mathbb{R}^d}\varphi)_- \geqslant \mathrm{Tr}(\varphi(\mathscr{L}_h^{\mathbb{R}^d})_-\varphi) - h^2\mathrm{Tr}(|\nabla\varphi|\gamma_h|\nabla\varphi|)\,.$$

With Proposition 2.4, we get

$$\mathrm{Tr}(\varphi\mathscr{L}_h^{\mathbb{R}^d}\varphi)_- \geqslant \mathrm{Tr}(\varphi(\mathscr{L}_h^{\mathbb{R}^d})_-\varphi) - Ch^{2-d}\|\nabla\varphi\|^2\,.$$

□

2.2. **Local traces on the boundary.** Let us consider here $\mathscr{L}_h^{\mathbb{R}^d_+} = -h^2\Delta - 1$ acting on $L^2(\mathbb{R}^d_+)$ with Dirichlet boundary condition on $x_d = 0$. Most of the statements of the previous section may be adapted to $\mathscr{L}_h^{\mathbb{R}^d_+}$, except, of course, the computation of the kernel.

Proposition 2.7. *For all* $\varphi \in \mathscr{C}_0^0(\mathbb{R}^d)$,

$$\mathrm{Tr}(\varphi(\mathscr{L}_h^{\mathbb{R}^d_+})_-\varphi) = 2(2\pi)^{-d}h^{-d}\int_{\mathbb{R}^d_+}\varphi^2(x)\int_{\mathbb{R}^d}(\xi^2-1)_-\sin^2(h^{-1}x_d\xi_d)\mathrm{d}\xi\mathrm{d}x\,.$$

Proof. Let us diagonalize $\mathscr{L}_h^{\mathbb{R}^d_+}$. For that purpose, let us consider the application $\mathscr{T}: L^2(\mathbb{R}^d_+)\to L^2(\mathbb{R}^d)$ defined by

$$\mathscr{T} = \frac{1}{\sqrt{2}}\mathscr{F}\circ S\,,$$

where S is defined by $S\psi(x) = \psi(x)$ when $x_d > 0$ and $S\psi(x) = -\psi(x)$ when $x_d \leqslant 0$. The operator $\mathscr{T}$ is an isometry and $\mathscr{T}: L^2(\mathbb{R}^d_+)\to L^2_{\mathrm{odd}}(\mathbb{R}^d)$ is bijective and $\mathscr{T}^{-1} = \sqrt{2}\mathscr{F}^{-1}$ where we have used $\mathscr{F}: L^2_{\mathrm{odd}}(\mathbb{R}^d)\to L^2_{\mathrm{odd}}(\mathbb{R}^d)$. We have

$$\mathscr{L}_h^{\mathbb{R}^d_+} = \mathscr{T}^{-1}(h^2|\xi|^2-1)\mathscr{T}\,.$$

In fact, $\mathscr{T}$ can be related to the "sine Fourier transform":

$$\mathscr{T}\psi(x) = \frac{1}{\sqrt{2}}(2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d}e^{-ix\xi}S\psi(x)\mathrm{d}x = -i\sqrt{2}(2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d_+}e^{-ix'\xi'}\sin(x_d\xi_d)\psi(x)\mathrm{d}x\,.$$

Notice that

$$(\mathscr{L}_h^{\mathbb{R}^d_+})_-^{\frac{1}{2}} = \mathscr{T}^{-1}(h^2|\xi|^2-1)_-^{\frac{1}{2}}\mathscr{T}\,.$$

In particular,

$$\begin{aligned}(\mathscr{L}_h^{\mathbb{R}^d_+})_-^{\frac{1}{2}}\psi(x) &= -\frac{2i}{(2\pi)^d}\int_{\mathbb{R}^d}e^{ix\xi}(h^2|\xi|^2-1)_-^{\frac{1}{2}}\int_{\mathbb{R}^d_+}e^{-iy'\xi'}\sin(y_d\xi_d)\psi(y)\mathrm{d}y\mathrm{d}\xi\\ &= -\frac{2i}{(2\pi)^d}\int_{\mathbb{R}^d_+}\mathrm{d}y\psi(y)\int_{\mathbb{R}^d}e^{ix\xi}e^{-iy'\xi'}\sin(y_d\xi_d)(h^2|\xi|^2-1)_-^{\frac{1}{2}}\mathrm{d}\xi\,.\end{aligned}$$

Thus, the kernel of $(\mathscr{L}_h^{\mathbb{R}^d_+})_-^{\frac{1}{2}}\varphi$ is

$$-\frac{2i}{(2\pi)^d}\varphi(y)\int_{\mathbb{R}^d} e^{ix\xi}e^{-iy'\xi'}\sin(y_d\xi_d)(h^2|\xi|^2-1)_-^{\frac{1}{2}}\mathrm{d}\xi\,.$$

The squared L^2-norm of this kernel is

$$\begin{aligned}
&\frac{4}{(2\pi)^{2d}}\int_{\mathbb{R}^d_+}\mathrm{d}y|\varphi(y)|^2\int_{\mathbb{R}^d_+}\mathrm{d}x\left|\int_{\mathbb{R}^d} e^{ix\xi}e^{-iy'\xi'}\sin(y_d\xi_d)(h^2|\xi|^2-1)_-^{\frac{1}{2}}\mathrm{d}\xi\right|^2\\
&=\frac{2}{(2\pi)^d}\int_{\mathbb{R}^d_+}\mathrm{d}y|\varphi(y)|^2\int_{\mathbb{R}^d}\mathrm{d}x\left|\int_{\mathbb{R}^d} e^{i(x-y)\xi}e^{iy_d\xi_d}\sin(y_d\xi_d)(h^2|\xi|^2-1)_-^{\frac{1}{2}}\mathrm{d}\xi\right|^2\\
&=\frac{2}{(2\pi)^d}\int_{\mathbb{R}^d_+}\mathrm{d}y|\varphi(y)|^2\int_{\mathbb{R}^d}\mathrm{d}\xi\sin^2(y_d\xi_d)(h^2|\xi|^2-1)_-\,.
\end{aligned}$$

This shows that $(\mathscr{L}_h^{\mathbb{R}^d_+})_-^{\frac{1}{2}}\varphi$ is a Hilbert-Schmidt operator, that $\varphi(\mathscr{L}_h^{\mathbb{R}^d_+})_-\varphi$ is trace-class, and

$$\mathrm{Tr}(\varphi(\mathscr{L}_h^{\mathbb{R}^d_+})_-\varphi)=\frac{2}{(2\pi)^d}\int_{\mathbb{R}^d_+}\mathrm{d}y|\varphi(y)|^2\int_{\mathbb{R}^d}\mathrm{d}\xi\sin^2(y_d\xi_d)(h^2|\xi|^2-1)_-\,.$$

□

In fact, one can estimate the asymptotic behavior of $\mathrm{Tr}(\varphi(\mathscr{L}_h^{\mathbb{R}^d_+})_-\varphi)$.

Lemma 2.8. *Let us consider the function defined for $t\geqslant 0$ by*

$$J(t)=\int_{\mathbb{R}^d}(\xi^2-1)_-\cos(2t\xi_d)\mathrm{d}\xi\,.$$

Then, for all $t\geqslant 0$,

$$J(t)=C_0\mathrm{Re}\,K(t)\,,\quad K(t)=\int_{-1}^{1}e^{2iut}(1-u^2)^{\frac{d+1}{2}}\mathrm{d}u\,,\quad C_0=\int_{\mathbb{B}^{d-1}}(1-|v|^2)\mathrm{d}v\,.$$

Moreover, $J(t)\underset{t\to+\infty}{=}\mathscr{O}(t^{-\frac{d+1}{2}-1})$.

Proof. We have

$$J(t)=\int_{-1}^{1}\mathrm{d}\xi_d\cos(2\xi_d t)\int_{|\xi'|^2\leqslant 1-|\xi_d|^2}(1-\xi_d^2-|\xi'|^2)\mathrm{d}\xi'\,.$$

By using a rescaling,

$$\int_{|\xi'|^2\leqslant 1-|\xi_d|^2}(1-\xi_d^2-|\xi'|^2)\mathrm{d}\xi'=(1-\xi_d^2)^{\frac{d+1}{2}}\int_{\mathbb{B}^{d-1}}(1-|v|^2)\mathrm{d}v\,.$$

Let us now consider K. We let $\delta=\frac{d+1}{2}$. By integrating by parts $\lfloor\delta\rfloor$ times, we can write

$$K(t)=t^{-\lfloor\delta\rfloor}\int_{-1}^{1}e^{2iut}k(u)(1-u^2)^{\delta-\lfloor\delta\rfloor}\mathrm{d}u\,,$$

where k is a polynomial.

If $\delta\in\mathbb{N}$, another integration by parts yields

$$K(t)=\mathscr{O}(t^{-\lfloor\delta\rfloor-1})\,,$$

which is the desired estimate.

If not, we have $\delta-\lfloor\delta\rfloor>0$ and we can integrate by parts:

$$K(t)=-it^{-\lfloor\delta\rfloor-1}\int_{-1}^{1}e^{2iut}uk(u)(1-u^2)^{\delta-\lfloor\delta\rfloor-1}\mathrm{d}u\,,$$

where we used that $(1-u^2)^{\delta-\lfloor\delta\rfloor-1}\in L^1((-1,1))$ since $-1<\delta-\lfloor\delta\rfloor-1<0$. Now, we can write, for some smooth function $\tilde{k}$,

$$\int_0^1 e^{2iut}uk(u)(1-u^2)^{\delta-\lfloor\delta\rfloor-1}\mathrm{d}u=\int_0^1 e^{2iut}\tilde{k}(u)(1-u)^{\delta-\lfloor\delta\rfloor-1}\mathrm{d}u\,,$$

and also, for some smooth function $\check{k}$,

$$\int_0^1 e^{2iut}uk(u)(1-u^2)^{\delta-\lfloor\delta\rfloor-1}\mathrm{d}u=e^{2it}\int_0^1 e^{-2ivt}\check{k}(v)v^{\delta-\lfloor\delta\rfloor-1}\mathrm{d}v\,.$$

Note that

$$\int_0^1 e^{-2ivt}\check{k}(v)v^{\delta-\lfloor\delta\rfloor-1}\mathrm{d}v$$
$$=\check{k}(0)\int_0^1 e^{-2ivt}v^{\delta-\lfloor\delta\rfloor-1}\mathrm{d}v+\int_0^1 e^{-2ivt}(\check{k}(v)-\check{k}(0))v^{\delta-\lfloor\delta\rfloor-1}\mathrm{d}v\,.$$

We have

$$\int_0^1 e^{-2ivt}v^{\delta-\lfloor\delta\rfloor-1}\mathrm{d}v=t^{\lfloor\delta\rfloor-\delta}\int_0^t e^{-2iv}v^{\delta-\lfloor\delta\rfloor-1}\mathrm{d}v=\mathscr{O}(t^{\lfloor\delta\rfloor-\delta})\,,$$

where we used that the last integral is convergent (by using integration by parts). We can write, for some smooth function r,

$$\check{k}(v)-\check{k}(0)=vr(v)\,,$$

so that

$$\int_0^1 e^{-2ivt}(\check{k}(v)-\check{k}(0))v^{\delta-\lfloor\delta\rfloor-1}\mathrm{d}v=\int_0^1 e^{-2ivt}r(v)v^{\delta-\lfloor\delta\rfloor}\mathrm{d}v=\mathscr{O}(t^{-1})=\mathscr{O}(t^{\lfloor\delta\rfloor-\delta})\,.$$

We deduce that

$$\int_0^1 e^{2iut}uk(u)(1-u^2)^{\delta-\lfloor\delta\rfloor-1}\mathrm{d}u=\mathscr{O}(t^{\lfloor\delta\rfloor-\delta})\,.$$

In the same way,

$$\int_{-1}^0 e^{2iut}uk(u)(1-u^2)^{\delta-\lfloor\delta\rfloor-1}\mathrm{d}u=\mathscr{O}(t^{\lfloor\delta\rfloor-\delta})\,.$$

Thus,

$$K(t)=\mathscr{O}(t^{-\lfloor\delta\rfloor-1+\lfloor\delta\rfloor-\delta})\,,$$

and the conclusion follows. ☐

Proposition 2.9. *Consider $\varphi\in\mathscr{C}_0^1(\mathbb{R}^d)$. Then,*

$$h^d\mathrm{Tr}(\varphi(\mathscr{L}_h^{\mathbb{R}^d_+})_-\varphi)=L_d\|\varphi\|^2_{L^2(\mathbb{R}^d_+)}-\frac{L_{d-1}}{4}h\int_{\mathbb{R}^{d-1}}|\varphi(x',0)|^2\mathrm{d}x'+\mathscr{O}(\ell^{d-1}h^2\|\nabla\varphi^2\|_\infty)\,.$$

Proof. Let us write

$$(2\pi h)^d\mathrm{Tr}(\varphi(\mathscr{L}_h^{\mathbb{R}^d_+})_-\varphi)=\int_{\mathbb{R}^d_+}\varphi^2(x)\int_{\mathbb{R}^d}(\xi^2-1)_-(1-\cos(2h^{-1}x_d\xi_d))\mathrm{d}\xi\mathrm{d}x\,.$$

Let us now consider the (absolutely convergent) integral

$$I(h)=\int_{\mathbb{R}^d_+}\varphi^2(x)\int_{\mathbb{R}^d}(\xi^2-1)_-\cos(2h^{-1}x_d\xi_d)\mathrm{d}\xi\mathrm{d}x\,.$$

We have

$$I(h)=h\int_{\mathbb{R}^d_+}\varphi^2(x',ht)\int_{\mathbb{R}^d}(\xi^2-1)_-\cos(2t\xi_d)\mathrm{d}\xi\mathrm{d}x'\mathrm{d}t\,, \tag{2.3}$$

and thus

$$I(h)=h\int_Q\left(\int_0^{+\infty}\varphi^2(x',ht)J(t)\mathrm{d}t\right)\mathrm{d}x'\,, \tag{2.4}$$

where Q is a compact subset of $\mathbb{R}^{d-1}$. We write, uniformly with respect to $x' \in Q$,

$$|\varphi^2(x', ht) - \varphi^2(x', 0)| \leqslant \|\nabla\varphi^2\|_\infty ht\,.$$

Therefore,

$$\left|\int_0^{+\infty} \varphi^2(x', ht)J(t)\mathrm{d}t - \int_0^{+\infty} \varphi^2(x', 0)J(t)\mathrm{d}t\right| \leqslant \|\nabla\varphi^2\|_\infty h \int_0^{+\infty} tJ(t)\mathrm{d}t\,.$$

This shows that

$$\left|I(h) - h\int_Q \mathrm{d}x' \int_0^{+\infty} \varphi^2(x', 0)J(t)\mathrm{d}t\right| \leqslant |Q|h^2\|\nabla\varphi^2\|_\infty \int_0^{+\infty} tJ(t)\mathrm{d}t\,.$$

We notice that

$$\int_{\mathbb{R}} (\xi^2 - 1)_- \cos(2t\xi_d)\mathrm{d}\xi_d = \mathrm{Re}\int_{\mathbb{R}} (\xi^2 - 1)_- e^{2it\xi_d}\mathrm{d}\xi_d\,.$$

Then, by using the inverse Fourier transform,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} (\xi^2 - 1)_- e^{2it\xi_d}\mathrm{d}\xi_d\mathrm{d}t = \frac{1}{2}(2\pi)(|\xi'|^2 - 1)_-\,.$$

Thus,

$$(2\pi)^{-d}\int_0^{+\infty} J(t)\mathrm{d}t = \frac{L_{d-1}}{4}\,.$$

□

3. Localization formula

Let us now start to explain how to glue the local trace estimates to estimate the global trace. For that purpose, one need to introduce a convenient partition of the unity. Consider $\chi \in \mathscr{C}_0^\infty(\mathbb{R}^d)$ supported in $B := B(0,1)$ and such that $\|\chi\|_{L^2} = 1$. Consider ℓ a smooth positive and bounded function such that $\|\nabla\ell\|_\infty =: \alpha < 1$.

Lemma 3.1. *Consider $F : \mathbb{R}^d \to \mathbb{R}^d$ defined by*

$$F(u) = \frac{x-u}{\ell(u)}\,.$$

Then, $F : \mathbb{R}^d \to \mathbb{R}^d$ is surjective. Moreover, letting $U := F^{-1}(B)$, the restriction $\tilde{F} : U \to B$ is a bijection.

Proof. Let us prove that F is surjective. Take $u \in \mathbb{R}^d$. For all $t \in \mathbb{R}$,

$$F(x - tu) = g(t)u\,, \quad g(t) = t\ell(x-tu)^{-1}\,.$$

Note that $g(0) = 0$ and $g(\ell_\infty) \geqslant 1$. By the intermediate value theorem, there exists $t_0 \in [0, \ell_\infty]$ such that $F(x - t_0u) = u$.

Let us prove that $\tilde{F}$ is injective. Consider $(u,v) \in U^2$ such that:

$$\ell(u)^{-1}(u-x) = \ell(v)^{-1}(v-x)\,.$$

We get

$$0 = \ell(v)(u-x) - \ell(u)(v-x) = (\ell(v) - \ell(u))(u-x) + \ell(u)(u-v)\,,$$

so that

$$\ell(u)|u-v| \leqslant \alpha|u-v||u-x| \leqslant \alpha\ell(u)|u-v|\,.$$

Thus, $u = v$. This shows that $\tilde{F}$ is injective, and thus bijective. □

Lemma 3.2. *Let $d\tilde{F}_u$ be the differential of $\tilde{F}$ at u. We have*

$$\forall h \in \mathbb{R}^d\,, \quad d\tilde{F}_u(h) = -\ell(u)^{-1}h - \ell(u)^{-2}(\nabla\ell(u)\cdot h)\,(x-u)\,.$$

Moreover,

$$\ell(u)^d|\det d\tilde{F}_u| = 1 + \ell(u)^{-1}\nabla\ell(u)\cdot(x-u)\,.$$

In particular, for all $u\in\Omega$, $\ell(u)^d|\det d\tilde{F}_u| \geqslant 1-\alpha > 0$, and $\tilde{F}$ is a smooth diffeomorphism.

Proof. We have

$$|\det(\ell(u)d\tilde{F}_u)| = \det(\mathrm{Id}+M)\,,$$

where $M = \ell(u)^{-1}[\partial_1\ell(u)(x-u),\ldots,\partial_d\ell(u)(x-u)]$. By using the multilinearity of the determinant, we see that

$$\det(\mathrm{Id}+M) = 1 + \ell(u)^{-1}\nabla\ell(u)\cdot(x-u)\,.$$

□

We let

$$J_u(x) = 1 + \ell(u)^{-1}(x-u)\cdot\nabla\ell(u)\,,$$

and

$$\phi_u(x) = \chi\left(\frac{x-u}{\ell(u)}\right)J_u(x)^{\frac{1}{2}}\,.$$

Lemma 3.3. *We have, for all $x\in\mathbb{R}^d$,*

$$\int_{\mathbb{R}^d}\phi_u(x)^2\ell(u)^{-d}\mathrm{d}u = 1\,.$$

Proof. By using the change of variable $\tilde{F}$,

$$\int_{\mathbb{R}^d}\phi_u(x)^2\ell(u)^{-d}\mathrm{d}u = \int_U\phi_u(x)^2\ell(u)^{-d}\mathrm{d}u = \int_B\chi\left(\frac{x-u}{\ell(u)}\right)^2 J_u(x)\ell(u)^{-d}\mathrm{d}u = 1\,.$$

□

Lemma 3.4. *Let $x\in\mathbb{R}^d$ and consider $u\in U$.*

$$(1-\alpha)\ell(u)\leqslant\ell(x)\leqslant(1+\alpha)\ell(u)\,.$$

Proof. We have

$$|\ell(u)-\ell(x)|\leqslant\alpha|u-x|\leqslant\alpha\ell(u)\,.$$

□

Lemma 3.5. *For all $\psi\in\mathrm{Dom}(\mathscr{Q}_h)$,*

$$\mathscr{Q}_h(\psi)\leqslant\int_{\mathbb{R}^d}\ell(u)^{-d}\mathscr{Q}_h(\phi_u\psi)\mathrm{d}u\,.$$

Moreover, there exists $C=C(\alpha,\chi)>0$ such that, for all $\psi\in\mathrm{Dom}(\mathscr{Q}_h)$,

$$\mathscr{Q}_h(\psi)\geqslant\int_{\mathbb{R}^d}\ell(u)^{-d}\mathscr{Q}_h(\phi_u\psi)\mathrm{d}u - Ch^2\int_{\mathbb{R}^{2d}}\ell(u)^{-d-2}|\phi_u\psi|^2\mathrm{d}x\mathrm{d}u\,.$$

Proof. We write the classical localization formula, for all $u\in\mathbb{R}^d$,

$$\langle\mathscr{L}_h\psi,\phi_u^2\psi\rangle = \mathscr{Q}_h(\phi_u\psi) - h^2\int_{\mathbb{R}^d}|\psi|^2|\nabla_x\phi_u|^2\mathrm{d}x\,,$$

and integrate

$$\int_{\mathbb{R}^d}\ell(u)^{-d}\langle\mathscr{L}_h\psi,\phi_u^2\psi\rangle\mathrm{d}u = \int_{\mathbb{R}^d}\ell(u)^{-d}\mathscr{Q}_h(\phi_u\psi)\mathrm{d}u - h^2\int_{\mathbb{R}^{2d}}|\psi|^2|\nabla_x\phi_u|^2\ell(u)^{-d}\mathrm{d}x\mathrm{d}u\,.$$

We have

$$\mathscr{Q}_h(\psi) = \int_{\mathbb{R}^d}\ell(u)^{-d}\mathscr{Q}_h(\phi_u\psi)\mathrm{d}u - h^2\int_{\mathbb{R}^{2d}}|\psi|^2|\nabla_x\phi_u|^2\ell(u)^{-d}\mathrm{d}x\mathrm{d}u\,.$$

Then,

$$\nabla_x \phi_u = J_u(x)^{\frac{1}{2}} \ell(u)^{-1} \nabla \chi \left(\frac{x-u}{\ell(u)}\right) + \frac{1}{2} \chi \left(\frac{x-u}{\ell(u)}\right) J_u^{-\frac{1}{2}} \nabla_x J_u \,.$$

On the support of ϕ_u,

$$1-\alpha \leqslant J_u \leqslant 1+\alpha \,. \tag{3.1}$$

Note also that

$$\nabla_x J_u = \ell(u)^{-1} \nabla \ell(u) \,.$$

It follows that

$$\begin{aligned}
|\nabla_x \phi_u|^2 &\leqslant C(\alpha)\ell(u)^{-2} \left(\left|\chi\left(\frac{x-u}{\ell(u)}\right)\right|^2 + \left|\nabla\chi\left(\frac{x-u}{\ell(u)}\right)\right|^2\right) \\
&\leqslant \tilde{C}(\alpha)\ell(x)^{-2} \left(\left|\chi\left(\frac{x-u}{\ell(u)}\right)\right|^2 + \left|\nabla\chi\left(\frac{x-u}{\ell(u)}\right)\right|^2\right) .
\end{aligned}$$

Let us integrate this last inequality with respect to u, use the change of variable $\tilde{F}$, and (3.1) to control the Jacobian,

$$\int_{\mathbb{R}^d} |\nabla_x \phi_u|^2 \ell(u)^{-d} \mathrm{d}u \leqslant \hat{C}(\alpha,\chi)\ell(x)^{-2} \,.$$

Thus,

$$\begin{aligned}
\int_{\mathbb{R}^{2d}} |\psi|^2 |\nabla_x \phi_u|^2 \ell(u)^{-d} \mathrm{d}x\mathrm{d}u &\leqslant \hat{C}(\alpha,\chi) \int_{\mathbb{R}^d} \ell(x)^{-2} |\psi|^2 \mathrm{d}x \\
&= \hat{C}(\alpha,\chi) \int_{\mathbb{R}^{2d}} \ell(x)^{-2} \ell(u)^{-d} |\phi_u \psi|^2 \mathrm{d}x\mathrm{d}u \\
&\leqslant \check{C}(\alpha,\chi) \int_{\mathbb{R}^{2d}} \ell(u)^{-2} \ell(u)^{-d} |\phi_u \psi|^2 \mathrm{d}x\mathrm{d}u \,.
\end{aligned}$$

□

4. From the local traces to the global trace

4.1. **From the local traces to the global trace.** The following proposition provides us with a lower bound of $\mathrm{Tr}(\mathscr{L}_h)_-$.

Proposition 4.1. *We have the following trace estimates.*

(i) *(Pointwise estimates) For all $u \in \mathbb{R}^2$,*

$$\mathrm{Tr}(\phi_u \mathscr{L}_h \phi_u)_- \leqslant \mathrm{Tr}(\phi_u (\mathscr{L}_h)_- \phi_u) = \mathrm{Tr}(\phi_u^2 (\mathscr{L}_h)_-) \,.$$

Moreover, for a Hilbert basis $(\psi_j)_{j\geqslant 1}$ adapted to the negative subspace of $\mathscr{L}_h$[1]*, we have*

$$\mathrm{Tr}(\phi_u^2 (\mathscr{L}_h)_-) = \sum_{j\in J} |\lambda_j| \|\phi_u \psi_j\|^2 \,.$$

(ii) *(Integrated estimates) In addition,*

$$\int_{\mathbb{R}^d} \mathrm{Tr}(\phi_u \mathscr{L}_h \phi_u)_- \ell(u)^{-d} \mathrm{d}u \leqslant \int_{\mathbb{R}^d} \mathrm{Tr}(\phi_u (\mathscr{L}_h)_- \phi_u) \ell(u)^{-d} \mathrm{d}u \leqslant \mathrm{Tr}(\mathscr{L}_h)_- \,.$$

Proof. The first item follows from the same analysis as in the proof of Corollary 2.5, and from the cyclicity of the trace. The second item follows from the Fubini-Tonelli theorem and

$$\int_{\mathbb{R}^d} \phi_u^2(x) \ell(u)^{-d} \mathrm{d}u = 1 \,.$$

□

Let us now consider the upper bound.

[1]We mean that there exists $J \subset \mathbb{N}^*$ such that $(\psi_j)_{j\in J}$ is a (Hilbert) basis of range$\mathbb{1}_{\lambda<0}(\mathscr{L}_h)$.

Proposition 4.2. *We have*

$$\mathrm{Tr}(\mathscr{L}_h)_- \leqslant \int_{\mathbb{R}^d} \ell(u)^{-d}\mathrm{Tr}(\phi_u \mathscr{L}_h \phi_u)_-\mathrm{d}u + Ch^{2-d}\int_{\Omega^{**}} \ell(x)^{-2}\mathrm{d}x\,,$$

where $\Omega^{**} = \cup_{u\in\Omega^*} B(u,\ell(u))$ *and* $\Omega^* = \{u \in \mathbb{R}^d : \mathrm{supp}\phi_u \cap \Omega \neq \emptyset\}$.

Proof. From Lemma 3.5,

$$\mathscr{Q}_h(\psi) \geqslant \int_{\mathbb{R}^d} \ell(u)^{-d}\mathscr{Q}_{h,u}(\phi_u\psi)\mathrm{d}u\,.$$

Consider $(\psi_j)_{j\geqslant 1}$ a Hilbert basis adapted to the negative subspace of $\mathscr{L}_h$. We have

$$\mathrm{Tr}(\mathscr{L}_h)_- = -\sum_{j\in J}\mathscr{Q}_h(\psi_j) \leqslant -\sum_{j\in J}\int_{\Omega^*}\ell(u)^{-d}\left(\mathscr{Q}_h(\phi_u\psi_j) - Ch^2\ell(u)^{-2}\|\phi_u\psi_j\|^2\right)\mathrm{d}u\,,$$

Consider $u\in\Omega^*$ and notice that, for all $\rho_u \in (0,1)$,

$$\begin{aligned}&\sum_{j\in J}\left(\mathscr{Q}_h(\phi_u\psi_j) - Ch^2\ell(u)^{-2}\|\phi_u\psi_j\|^2\right)\\ &\qquad= \sum_{j\in J}(1-\rho_u)\mathscr{Q}_h(\phi_u\psi_j) + \sum_{j\in J}\left(\rho_u\mathscr{Q}_h(\phi_u\psi_j) - Ch^2\ell(u)^{-2}\|\phi_u\psi_j\|^2\right)\,.\end{aligned}$$

This can rewritten as

$$\begin{aligned}&\sum_{j\in J}\left(\mathscr{Q}_h(\phi_u\psi_j) - Ch^2\ell(u)^{-2}\|\phi_u\psi_j\|^2\right)\\ &\qquad= (1-\rho_u)\mathrm{Tr}(\phi_u\mathscr{L}_h\phi_u\Pi_h^-) + \mathrm{Tr}(\phi_u\left(\rho_u\mathscr{L}_h - Ch^2\ell(u)^{-2}\right)\phi_u\Pi_h^-)\,.\end{aligned}$$

where $\Pi_h^- = \mathbb{1}_{\lambda<0}(\mathscr{L}_h)$ (which has finite rank). From the Variational Principle, we have

$$\begin{aligned}&-\sum_{j\in J}\left(\mathscr{Q}_h(\phi_u\psi_j) - Ch^2\ell(u)^{-2}\|\phi_u\psi_j\|^2\right)\\ &\leqslant (1-\rho_u)\mathrm{Tr}\left(\phi_u\mathscr{L}_h\phi_u\right)_- + \mathrm{Tr}\left(\phi_u\left(\rho_u\mathscr{L}_h - Ch^2\ell(u)^{-2}\right)\phi_u\right)_-\\ &\leqslant \mathrm{Tr}\left(\phi_u\mathscr{L}_h\phi_u\right)_- + \mathrm{Tr}\left(\phi_u\left(\rho_u\mathscr{L}_h - Ch^2\ell(u)^{-2}\right)\phi_u\right)_-\,.\end{aligned}$$

Let us take $\rho_u = Ch^2\ell(u)^{-2}$ and notice that

$$\mathrm{Tr}\left(\phi_u\left(\rho_u\mathscr{L}_h - Ch^2\ell(u)^{-2}\right)\phi_u\right)_- = Ch^2\ell(u)^{-2}\mathrm{Tr}\left(\phi_u\left(-h^2\Delta - 2\right)\phi_u\right)_-\,.$$

Since

$$\mathrm{Tr}\left(\phi_u\left(-h^2\Delta-2\right)\phi_u\right)_- \leqslant \mathrm{Tr}\left(\phi_u\left(-h^2\Delta-2\right)_-\phi_u\right) \leqslant Ch^{-d}\|\phi_u\|^2\,,$$

we deduce that

$$\mathrm{Tr}(\mathscr{L}_h)_- \leqslant \int_{\mathbb{R}^d}\ell(u)^{-d}\mathrm{Tr}(\phi_u\mathscr{L}_h\phi_u)_-\mathrm{d}u + Ch^{2-d}\int_{\Omega^*}\int_{\mathbb{R}^d}\ell(u)^{-d-2}\phi_u^2(x)\mathrm{d}x\mathrm{d}u\,.$$

Since

$$\int_{\Omega^*}\int_{\mathbb{R}^d}\ell(u)^{-d-2}\phi_u^2(x)\mathrm{d}x\mathrm{d}u \leqslant C\int_{\Omega^*}\int_{\Omega^{**}}\ell(x)^{-2}\ell(u)^{-d}\phi_u^2(x)\mathrm{d}x\mathrm{d}u \leqslant C\int_{\Omega^{**}}\ell(x)^{-2}\mathrm{d}x\,.$$

∎

Therefore, we must estimate the local trace $\mathrm{Tr}(\phi_u\mathscr{L}_h\phi_u)_-$.

4.2. **Estimates of the local traces.** Let us consider

$$\mathscr{I}(h) = \int_{\Omega^*} \ell(u)^{-d}\mathrm{Tr}(\phi_u \mathscr{L}_h \phi_u)_- \mathrm{d}u\,, \tag{4.1}$$

and write

$$\mathscr{I}(h) = \int_{U_1} \ell(u)^{-d}\mathrm{Tr}(\phi_u \mathscr{L}_h \phi_u)_- \mathrm{d}u + \int_{U_2} \ell(u)^{-d}\mathrm{Tr}(\phi_u \mathscr{L}_h \phi_u)_- \mathrm{d}u\,, \tag{4.2}$$

where $U_1 = \{u \in \Omega^* : \mathrm{supp}\phi_u \cap \partial\Omega = \emptyset\}$ and $U_2 = \Omega^* \setminus U_1$.

4.2.1. *Inside balls.* From Proposition 2.6 and Corollary 2.5, we have, for all $u \in U_1$,

$$\begin{aligned}\mathrm{Tr}(\phi_u \mathscr{L}_h \phi_u)_- &= \mathrm{Tr}(\phi_u (\mathscr{L}_h)_- \phi_u) + \mathscr{O}(h^{2-d}\ell(u)^{d-2}) \\ &= h^{-d} L_d \|\phi_u\|^2_{L^2(\Omega)} + \mathscr{O}(h^{2-d}\ell(u)^{d-2})\,.\end{aligned} \tag{4.3}$$

4.2.2. *Boundary coordinates.* Considering $u \in U_2$, one will need boundary coordinates.

Given a ball B (small enough) such that $B \cap \partial\Omega \neq \emptyset$, one can consider $x_0 \in B \cap \partial\Omega$. By assumption, the boundary can be locally described as the graph of a $\mathscr{C}^{1,\alpha}$ function:

$$B \cap \partial\Omega = \{(x', f(x'))\,, x' \in D\}\,.$$

We can even assume that $x_0 = 0$, $f(0) = 0$, and $\nabla_{x'} f(0) = 0$ so that

$$\sup_{x' \in D} |\nabla_{x'} f(x')| \leqslant C\ell^\alpha\,.$$

This description of the boundary allows to define new coordinates $y = \kappa(x)$:

$$\forall j \in \{1, \dots, d-1\}\,, \quad y_j = x_j\,, \qquad y_d = x_d - f(x')\,.$$

The map $\kappa : D \times \mathbb{R} \to D \times \mathbb{R}$ is a local $\mathscr{C}^1$-diffeomorphism whose Jacobian is 1. It sends $B \cap \partial\Omega$ onto $D \times \{0\}$.

Now, if $\phi \in \mathscr{C}_0^\infty(B)$, we let

$$\tilde{\phi} = \phi \circ \kappa^{-1}\,,$$

which is extended by 0 to $\mathbb{R}^d$.

4.2.3. *Boundary balls.*

Lemma 4.3. *We have*

$$|\mathrm{Tr}(\phi_u \mathscr{L}_h \phi_u)_- - \mathrm{Tr}(\tilde{\phi}_u \mathscr{L}_h^{\mathbb{R}^d_+} \tilde{\phi}_u)_-| \leqslant C\ell(u)^{\alpha+d} h^{-d}\,,$$

and

$$\int_\Omega \phi_u^2 \mathrm{d}x = \int_{\mathbb{R}^d_+} \tilde{\phi}_u^2(y)\mathrm{d}y\,,$$

$$\int_{\partial\Omega} \phi_u^2 \mathrm{d}\sigma = \int_{\mathbb{R}^{d-1}} \tilde{\phi}_u^2(x', 0)\mathrm{d}x' + \mathscr{O}(\ell(u)^{d-1+2\alpha})\,.$$

Proof. We have, by definition of the surface measure,

$$\int_{\partial\Omega} \phi_u^2 \mathrm{d}\sigma = \int_{\mathbb{R}^{d-1}} \sqrt{1 + |\nabla f(x')|^2}\,\tilde{\phi}_u^2(x', 0)\mathrm{d}x'\,.$$

Then, we set $\Psi = \phi_u \psi \in H_0^1(\Omega)$ and write, in the sense of quadratic forms,

$$\langle -h^2 \Delta^\Omega \Psi, \Psi\rangle = (1 + \mathscr{O}(\ell(u)^\alpha))\langle -h^2 \Delta^{\mathbb{R}^d_+} \tilde{\Psi}, \tilde{\Psi}\rangle\,.$$

In particular,

$$\langle -h^2 \Delta^\Omega \Psi, \Psi\rangle \geqslant (1 - C\ell(u)^\alpha)\langle -h^2 \Delta^{\mathbb{R}^d_+} \tilde{\Psi}, \tilde{\Psi}\rangle\,.$$

By definition,

$$\mathrm{Tr}(\phi_u \mathscr{L}_h \phi_u)_- = -\sum_{j \in J^-} \mathscr{Q}_h(\phi_u \psi_j) \leqslant -\sum_{j \in J^-} \left((1 - C\ell(u)^\alpha)\langle -h^2 \Delta^{\mathbb{R}^d_+} \tilde{\Psi}_j, \tilde{\Psi}_j\rangle - \|\tilde{\Psi}_j\|^2\right)\,.$$

Then,

$$\mathrm{Tr}(\phi_u \mathscr{L}_h \phi_u)_- \leqslant -\sum_{j\in J^-}\left((1-C\ell(u)^\alpha)\langle -h^2\Delta^{\mathbb{R}^d_+}\tilde{\Psi}_j,\tilde{\Psi}_j\rangle - \|\tilde{\Psi}_j\|^2\right)\,,$$

so that

$$\mathrm{Tr}(\phi_u \mathscr{L}_h \phi_u)_- \leqslant -\sum_{j\in J^-}(1-2C\ell(u)^\alpha)\left[\langle -h^2\Delta^{\mathbb{R}^d_+}\tilde{\Psi}_j,\tilde{\Psi}_j\rangle - \|\tilde{\Psi}_j\|^2\right]$$
$$+ C\ell(u)^\alpha\left[\langle -h^2\Delta^{\mathbb{R}^d_+}\tilde{\Psi}_j,\tilde{\Psi}_j\rangle - 2\|\tilde{\Psi}_j\|^2\right]\,.$$

From the Variational Principle, it follows that

$$\mathrm{Tr}(\phi_u \mathscr{L}_h \phi_u)_- \leqslant \mathrm{Tr}(\tilde{\phi}_u \mathscr{L}_h^{\mathbb{R}^d_+}\tilde{\phi}_u)_- + C\ell(u)^\alpha h^{-d}\ell(u)^d\,.$$

Exchanging the roles of the two quadratic forms, the conclusion follows. □

Lemma 4.4. *For $u\in U_2$, we have*

$$\left|\mathrm{Tr}(\phi_u \mathscr{L}_h \phi_u)_- - \left(h^{-d}L_d\|\phi_u\|^2_{L^2(\Omega)} - h^{-d+1}\frac{L_{d-1}}{4}\|\phi_u\|^2_{L^2(\partial\Omega)}\right)\right|$$
$$\leqslant C\ell(u)^{\alpha+d}h^{-d} + C\ell(u)^{d-1+2\alpha}h^{-d+1} + C\ell(u)^{d-2}h^{2-d}\,.$$

Proof. It is enough to combine Lemma 4.3 and Proposition 2.9. □

5. Proof of the two-term asymptotics

Finally, we can prove Theorem 1.1.

5.1. **Gathering the estimates.** From Propositions 4.1 and 4.2, we have, with (4.1),

$$\mathscr{I}(h) \leqslant \mathrm{Tr}(\mathscr{L}_h)_- \leqslant \mathscr{I}(h) + Ch^{2-d}\int_{\Omega^{**}}\ell(x)^{-2}\mathrm{d}x\,.$$

We use the splitting (4.2), the estimate (4.3), and Lemma 4.4 to deduce, via Lemma 3.3,

$$\left|\mathrm{Tr}(\mathscr{L}_h)_- - \left(h^{-d}L_d|\Omega| - h^{-d+1}\frac{L_{d-1}}{4}|\partial\Omega|\right)\right|$$
$$\leqslant Ch^{2-d}\int_{\Omega^{**}}\mathrm{d}u\,\ell(u)^{-2} + C\int_{U_2}\mathrm{d}u\left(\ell(u)^\alpha h^{-d} + \ell(u)^{-1+2\alpha}h^{-d+1}\right)\,.$$

5.2. **Estimating the remainders.** At this stage of the analysis, we have to make a choice of $\ell(u)$. Consider

$$\ell(u) = \frac{1}{2}\sqrt{\ell_0^2 + \mathrm{dist}^2(u,\partial\Omega)}\,,$$

for some $\ell_0 > 0$ small enough to be determined. This choice fulfills all the requirements of Section 3. We have (by using coordinates in a fixed neighborhood of $\partial\Omega$)

$$\int_{\Omega^{**}}\mathrm{d}u\,\ell(u)^{-2} \leqslant C\ell_0^{-1}\,.$$

In addition, for all $u\in U_2$, we have $\ell(u) > \mathrm{dist}(u,\partial\Omega)$ and this is equivalent to

$$\mathrm{dist}(u,\partial\Omega) < \frac{1}{\sqrt{3}}\ell_0\,,$$

so that

$$\frac{\ell_0}{2} \leqslant \ell(u) < \frac{2}{\sqrt{3}}\ell_0\,.$$

We deduce

$$\int_{U_2} \mathrm{d}u \left(\ell(u)^\alpha h^{-d} + \ell(u)^{-1+2\alpha} h^{-d+1}\right) \leqslant Ch^{-d}\ell_0^{1+\alpha} + Ch^{-d+1}\ell_0^{2\alpha}\,.$$

It remains to choose ℓ_0 in order to optimize the remainders:

$$h^{2-d}\ell_0^{-1} + h^{1-d}\ell_0^{2\alpha} + h^{-d}\ell_0^{1+\alpha}\,.$$

We choose $\ell_0 = h^{\frac{2}{2+\alpha}}$ and the remainders become

$$h^{-d}(h^{1+\frac{4\alpha}{2+\alpha}} + h^{\frac{2+2\alpha}{2+\alpha}}) = o(h^{1-d})\,.$$

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