Magnetic Harmonic Approximation

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This talk is devoted to the magnetic Laplacian

$$\boxed{\mathscr{L}_\hbar = (-i\hbar\nabla - \mathbf{A})^2}\,,$$

- acting on $L^2(\Omega)$ with $\Omega \subset \mathbb{R}^d$,
- with $\mathbf{A} : \overline{\Omega} \to \mathbb{R}^d$,
- with some boundary conditions on $\partial\Omega$.

A question to the audience

Is the magnetic Laplacian elliptic?

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ii. classical mechanics of charged particles submitted to magnetic fields and its quantization,

iii. analogy between the electric Laplacian $-\hbar^2\Delta + V$ and the magnetic Laplacian $(-i\hbar\nabla - \mathbf{A})^2$.

Electric Harmonic Approximation

Let us recall what the harmonic approximation is. If the electric potential V admits a unique and non degenerate minimum (not attained at infinity), then the m-th eigenvalue $\lambda_m(\hbar)$ satisfies

$$\boxed{\lambda_m(\hbar) = V(x_{\min}) + \mu_m \hbar + o(\hbar)}\,,$$

where μ_m is the m-th eigenvalue of $D_x^2 + \frac{1}{2}\mathrm{Hess}_{x_{\min}} V(x)$.

What are magnetic fields?

The magnetic 1-form is

$$\alpha = \sum_{j=1}^{d} A_k \mathrm{d}x_k$$

and the magnetic 2-form $\mathrm{d}\alpha$ is identified with

$$[d=2] \quad B = \partial_1 A_2 - \partial_2 A_1,$$

$$[d=3] \quad \mathbf{B} = (\partial_2 A_3 - \partial_3 A_2, \partial_3 A_1 - \partial_1 A_3, \partial_1 A_2 - \partial_2 A_1).$$

About $\lambda_1(\hbar)$

Until 2006, the main motivation was about estimating the third critical field in the Ginzburg-Landau theory (see the book by **Fournais-Helffer**). There were many contributions (**Bauman-Philips-Tang**, **Bolley-Helffer**, **Erdös**, etc.). These works aimed at estimating one or two terms in the semiclassical expansion of $\lambda_1(\hbar)$.

An example of asymptotic result

Among a vast literature, let us pick up one result.

Theorem (Helffer-Morame)

Assume that $\Omega \subset \mathbb{R}^2$ is smooth and bounded. Assume also that $B = 1$ and that the boundary carries the magnetic Neumann boundary condition. Then

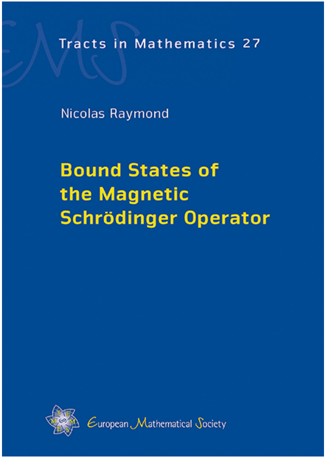
$$\lambda_1(\hbar) = \Theta_0 \hbar - C_1 \kappa_{\max} \hbar^{\frac{3}{2}} + o(\hbar^{\frac{3}{2}}) .$$

A similar theorem has been proved by the same authors in three dimensions. It involves much advanced geometric considerations and a “magnetic curvature”.

What about $\lambda_2(\hbar)$?

Is $\lambda_1(\hbar)$ simple?

Propaganda



Bound States of the Magnetic Schrödinger Operator
EMS Tracts (27) (2017).

Until 2009, there were only three results concerned with the other eigenvalues, in two dimensions:

- when $B = 1$, Ω bounded and smooth, magnetic Neumann condition, see **Fournais-Helffer**,
- when B has a unique and non-degenerate minimum, see **Helffer-Kordyukov**,
- when Ω is a corner domain, see **Bonnaillie-Noël–Dauge**.

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The methods used to prove these results strongly differ.

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- in the presence of a conical singularity (with **Bonnaillie-Noël–R.**).

We have established that, in all these situations, the magnetic Laplacian is microlocally and unitarily equivalent to an pure electric Laplacian in an "adiabatic form".

In fact, it is not always possible to make such a reduction:

- in the case of non-vanishing variable magnetic fields in $\mathbb{R}^d$ (**Helffer-Kordyukov**),
- in the case certain vanishing magnetic fields (**Dauge-Miqueu-R.**),
- in cases with corners (**Bonnaillie-Noël–Dauge–Popoff**, groundenergy).

Electric Born-Oppenheimer approximation

Consider

$$-h^2\Delta_s - \Delta_t + V(s,t) \,.$$

The main idea, due to Born and Oppenheimer, is to replace, for fixed s, the operator $-\Delta_t + V(s,t)$ by its eigenvalues $\mu_k(s)$. Then we are led to consider for instance the reduced operator

$$-h^2\Delta_s + \mu_1(s) \,,$$

and to apply the semiclassical techniques *à la* Helffer-Sjöstrand.

Quantum averaging

The idea is to find $P_\hbar$ such that

$$[P_\hbar, \mathscr{L}_\hbar] = \mathscr{O}(\hbar^n)\,.$$

We look at this projection in the form

$$P_\hbar = \mathrm{Op}_\hbar^{\mathrm{W}}\left(\Pi_{x,\xi,\hbar}\right)\,, \qquad \text{where } x \text{ is the effective semiclassical variable}\,.$$

See **Jecko**, **Martinez-Sordoni**, **Panati-Spohn-Teufel** where such ideas are developed in the context of quantum evolution.

A partially semiclassical magnetic Laplacian

The investigation of magnetic Laplacians leads to the self-adjoint operators on the space $\mathsf{L}^2(\mathbb{R}^m_s \times \mathbb{R}^n_t, \mathrm{d}s\mathrm{d}t)$ of the following type

$$\boxed{\mathfrak{L}_h = (hD_s - A_1(s,t))^2 + (D_t - A_2(s,t))^2}\,,$$

where A_1 and A_2 are polynomials.

Let us write the operator valued symbol of $\mathfrak{L}_h$. For $(x, \xi) \in \mathbb{R}^m \times \mathbb{R}^m$, we introduce the electro-magnetic Laplacian acting on $\mathsf{L}^2(\mathbb{R}^n, \mathrm{d}t)$:

$$\boxed{\mathcal{M}_{x,\xi} = (D_t - A_2(x,t))^2 + (\xi - A_1(x,t))^2} .$$

Denoting by $\mu(x, \xi)$ its lowest eigenvalue we would like to replace $\mathfrak{L}_h$ by the m-dimensional pseudo-differential operator:

$$\mu(s, hD_s).$$

Assumption 1

Assumption

- *The family* $(\mathcal{M}_{x,\xi})_{(x,\xi)\in\mathbb{R}^m\times\mathbb{R}^m}$ *is analytic of type (B) in the sense of Kato.*
- *For all* $(x,\xi)\in\mathbb{R}^m\times\mathbb{R}^m$, *the bottom of the spectrum of* $\mathcal{M}_{x,\xi}$ *is a simple eigenvalue denoted by* $\mu(x,\xi)$ *(in particular it is an analytic function) and associated with a* L^2*-normalized eigenfunction* $u_{x,\xi}\in\mathcal{S}(\mathbb{R}^n)$ *which also analytically depends on* (x,ξ).
- *The function* μ *admits a unique and non degenerate minimum* μ_0 *at point denoted by* (x_0,ξ_0) *and* $\liminf_{|x|+|\xi|\to+\infty}\mu(x,\xi)>\mu_0$.
- *The family* $(\mathcal{M}_{x,\xi})_{(x,\xi)\in\mathbb{R}^m\times\mathbb{R}^m}$ *can be analytically extended in a complex neighborhood of* (x_0,ξ_0).

Assumption 2

Assumption

Under the last assumption , let us denote by $\operatorname{Hess}\mu(x_0,\xi_0)$ *the Hessian matrix of* μ *at* (x_0,ξ_0). *We assume that the spectrum of the operator* $\operatorname{Hess}\mu(x_0,\xi_0)(\sigma,D_\sigma)$ *is simple.*

Asymptotic expansions of $\lambda_n(h)$

Theorem (Bonnaillie-Noël–Hérau–R.)

For all $n \geq 1$, there exists $h_0 > 0$ such that for all $h \in (0, h_0)$ the n-th eigenvalue of $\mathfrak{L}_h$ exists and satisfies

$$\lambda_n(h) = \lambda_{n,0} + \lambda_{n,1} h + \mathcal{O}(h^{\frac{3}{2}}) ,$$

$\lambda_{n,0} = \mu_0$ *and* $\lambda_{n,1}$ *is the n-th eigenvalue of* $\frac{1}{2}\mathrm{Hess}_{x_0,\xi_0}\,\mu(\sigma, D_\sigma)$.

In concrete situations the term $\lambda_{n,1}$ involves a curvature term. Generalizations appear in the thesis of **Keraval** (Weyl laws).

Flavor of the proof

Let us recall the formalism of coherent states. We define

$$g_0(\sigma) = \pi^{-m/4} \mathrm{e}^{-|\sigma|^2/2},$$

and the usual creation and annihilation operators

$$\mathfrak{a}_j = \frac{1}{\sqrt{2}}(\sigma_j + \partial_{\sigma_j}), \qquad \mathfrak{a}_j^* = \frac{1}{\sqrt{2}}(\sigma_j - \partial_{\sigma_j}),$$

which satisfy the commutator relations

$$[\mathfrak{a}_j, \mathfrak{a}_j^*] = 1, \qquad [\mathfrak{a}_j, \mathfrak{a}_k^*] = 0 \quad \text{if } k \neq j.$$

We notice that

$$\sigma_j = \frac{1}{\sqrt{2}}(\mathfrak{a}_j + \mathfrak{a}_j^*), \qquad \partial_{\sigma_j} = \frac{1}{\sqrt{2}}(\mathfrak{a}_j - \mathfrak{a}_j^*), \qquad \mathfrak{a}_j\mathfrak{a}_j^* = \frac{1}{2}(D_{\sigma_j}^2 + \sigma_j^2 + 1)\,.$$

For $(u, p) \in \mathbb{R}^m \times \mathbb{R}^m$, we introduce the coherent state

$$f_{u,p}(\sigma) = \mathrm{e}^{ip\cdot\sigma} g_0(\sigma - u)\,,$$

and the associated projection, defined for $\psi \in \mathsf{L}^2(\mathbb{R}^m \times \mathbb{R}^n)$ by

$$\Pi_{u,p}\psi = \langle \psi, f_{u,p}\rangle_{\mathsf{L}^2(\mathbb{R}^m,\mathrm{d}\sigma)} f_{u,p} = \psi_{u,p} f_{u,p}\,,$$

which satisfies

$$\psi = \int_{\mathbb{R}^{2m}} \Pi_{u,p}\psi \,\mathrm{d}u\mathrm{d}p\,,$$

and the Parseval formula

$$\|\psi\|^2 = \int_{\mathbb{R}^n}\int_{\mathbb{R}^{2m}} |\psi_{u,p}|^2 \mathrm{d}u\mathrm{d}p\mathrm{d}\tau\,.$$

We recall that

$$(\mathfrak{a}_j)^\ell(\mathfrak{a}_k^*)^q\psi = \int_{\mathbb{R}^{2m}} \left(\frac{u_j + ip_j}{\sqrt{2}}\right)^\ell \left(\frac{u_k - ip_k}{\sqrt{2}}\right)^q \Pi_{u,p}\psi \mathrm{d}u\mathrm{d}p.$$

The rescaled operator ($s = x_0 + h^{1/2}\sigma,\ t = \tau$) is

$$\mathcal{L}_h = \left(D_\tau + A_2(x_0 + h^{1/2}\sigma, \tau)\right)^2 + \left(\xi_0 + h^{1/2}D_\sigma + A_1(x_0 + h^{1/2}\sigma, \tau)\right)^2$$

and

$$\mathcal{L}_h = \mathcal{L}_0 + h^{1/2}\mathcal{L}_1 + h\mathcal{L}_2 + \ldots + h^{M/2}\mathcal{L}_M.$$

If we write the anti-Wick ordered operator, we get

$$\mathcal{L}_h = \underbrace{\mathcal{L}_0 + h^{1/2}\mathcal{L}_1 + h\mathcal{L}_2^{\mathrm{W}} + \ldots + (h^{1/2})^M \mathcal{L}_M^{\mathrm{W}}}_{\mathcal{L}_h^{\mathrm{W}}} + \underbrace{hR_2 + \ldots + (h^{1/2})^M R_M}_{\mathcal{R}_h},$$

where the R_d are the remainders in the anti-Wick ordering and satisfy, for $d \geq 2$,

$$h^{d/2} R_d = h^{d/2}\mathcal{O}_{d-2}(\sigma, D_\sigma),$$

where the notation $\mathcal{O}_d(\sigma, D_\sigma)$ stands for a polynomial operator with total degree in (σ, D_σ) less than d. We recall that

$$\boxed{\mathcal{L}_h^{\mathrm{W}} = \int_{\mathbb{R}^{2m}} \mathcal{M}_{x_0+h^{1/2}u,\xi_0+h^{1/2}p}\,\mathrm{d}u\mathrm{d}p} \,.$$

We reduce here our study to the case when $A_2 = 0$. We therefore focus now on operators of the form

$$\mathfrak{L}_h = D_t^2 + (hD_s + A_1(s,t))^2.$$

Theorem (**Bonnaillie-Noël–Hérau–R.**)

Under our assumptions, there exist a function $\Phi = \Phi(s)$ *defined in a neighborhood* $\mathcal{V}$ *of* x_0 *with* $\mathrm{Re}\,\mathsf{Hess}\,\Phi(x_0) > 0$ *and, for any* $n \geq 1$*, a sequence of real numbers* $(\lambda_{n,j})_{j\geq 0}$ *such that*

$$\lambda_n(h) \underset{h\to 0}{\sim} \sum_{j\geq 0} \lambda_{n,j} h^j\,,$$

with $\lambda_{n,0} = \mu_0$*. Besides there exists a formal series of smooth functions on* $\mathcal{V} \times \mathbb{R}^n_t$*,*

$$a_n(.;h) \underset{h\to 0}{\sim} \sum_{j\geq 0} a_{n,j} h^j, \quad \textit{with } a_{n,0} \neq 0 \textit{ such that}$$

$$\left(\mathfrak{L}_h - \lambda_n(h)\right)\left(a_n(.;h)e^{-\Phi/h}\right) = \mathcal{O}\left(h^\infty\right) e^{-\Phi/h}\,.$$

In addition, there exists $c_0 > 0$ *such that for all* $h \in (0, h_0)$

$$\mathcal{B}\Big(\lambda_{n,0} + \lambda_{n,1} h, c_0 h\Big) \cap \mathsf{sp}\left(\mathfrak{L}_h\right) = \left\{\lambda_n(h)\right\}.$$

Thanks to our theorem giving the splitting of the lowest eigenvalues, we have sharp asymptotic expansions of the eigenvalues. In particular, one knows that they become simple in the semiclassical limit and we get the approximation of the eigenfunctions by the WKB expansions.

Flavor of the proof

We write

$$\mathfrak{L}_h^\natural = D_t^2 + (hD_s + A^\natural)^2, \qquad A^\natural(s,t) = \xi_0 + A_1(x_0 + s, t)\,.$$

In order to lighten the notation, we introduce

$$\mathcal{M}^\natural_{x,\xi} = \mathcal{M}_{x+x_0,\xi+\xi_0}\,, \quad u^\natural_{x,\xi} = u_{x+x_0,\xi+\xi_0}\,, \quad \mu^\natural(x,\xi) = \mu(x+x_0,\xi+\xi_0)\,.$$

We have

$$\left(\mathcal{M}^\natural_{x,\xi}\right)^* = \mathcal{M}^\natural_{x,\overline{\xi}}\,, \qquad \forall x \in \mathbb{R}^m\,, \forall \xi \in \mathbb{C}^m\,.$$

The assumption $A_2 = 0$ implies the fundamental property:

$$\overline{u^\natural_{x,\xi}} = u^\natural_{x,\overline{\xi}}\,.$$

We conjugate $\mathfrak{L}_h^\natural$ via a weight function $\Phi = \Phi(s)$ and define

$$\begin{aligned}\mathfrak{L}_\Phi^\natural &= \mathrm{e}^{\Phi(s)/h}\, \mathfrak{L}_h^\natural\, \mathrm{e}^{-\Phi(s)/h}\\ &= D_t^2 + (hD_s + i\nabla\Phi + A^\natural)^2\\ &= \mathfrak{L}_0^\natural + h\mathfrak{L}_1^\natural + h^2\mathfrak{L}_2^\natural\,,\end{aligned}$$

with

$$\mathfrak{L}_0^\natural = D_t^2 + (i\nabla\Phi + A^\natural)^2 = \mathcal{M}^\natural_{s,i\nabla\Phi(s)}\,,$$

$$\mathfrak{L}_1^\natural = \frac{1}{2}\left(D_s \cdot (\nabla_\xi\mathcal{M}^\natural)_{s,i\nabla\Phi(s)} + (\nabla_\xi\mathcal{M}^\natural)_{s,i\nabla\Phi(s)} \cdot D_s\right),$$

$$\mathfrak{L}_2^\natural = D_s^2\Phi\,.$$

We now look for a formal solution in the form

$$\lambda \sim \sum_{j\geq 0} \lambda_j h^j, \qquad a \sim \sum_{j\geq 0} a_j h^j$$

such that $\mathfrak{L}^\natural_\Phi a = \lambda a$.

We have to find (λ_0, a_0) such that

$$\mathfrak{L}_0^\natural a_0 = \lambda_0 a_0 \,.$$

We must choose

$$\lambda_0 = \mu_0 \,.$$

Thus we have to find a_0 such that

$$\mathcal{M}^\natural_{s, i\nabla\Phi(s)} a_0 = \mu_0 a_0 \,.$$

We choose a_0 in the form

$$\boxed{a_0(s,t) = u^\natural_{s, i\nabla\Phi(s)}(t) b_0(s)} \,,$$

where b_0 has to be determined and Φ is the solution of the following eikonal equation (justified by our analyticity assumptions)

$$\boxed{\mu^\natural(s, i\nabla_s\Phi) = \mu_0} \,.$$

Collecting the terms in h^1, we obtain the first transport equation

$$(\mathfrak{L}_0^\natural - \mu_0)a_1 = -(\mathfrak{L}_1^\natural - \lambda_1)a_0\,.$$

Pointwise in s, the Fredholm compatibility condition writes

$$(\lambda_1 - \mathfrak{L}_1^\natural)a_0 \in (\mathrm{Ker}(\mathfrak{L}_0^{\natural\,*} - \mu_0))^\perp\,.$$

We have $\mathrm{Ker}(\mathfrak{L}_0^{\natural\,*} - \mu_0) = \mathrm{span}(u^\natural_{s,-i\nabla\overline{\Phi}(s)})$, so that the compatibility condition is equivalent to

$$\boxed{\lambda_1 \left\langle u^\natural_{s,i\nabla\Phi(s)} b_0(s), u^\natural_{s,-i\nabla\overline{\Phi}(s)} \right\rangle_{\mathsf{L}^2(\mathbb{R}^m,\mathrm{d}t)} = \left\langle \mathfrak{L}_1^\natural u^\natural_{s,i\nabla\Phi(s)} b_0(s), u^\natural_{s,-i\nabla\Phi(s)} \right\rangle_{\mathsf{L}^2(\mathbb{R}^m,\mathrm{d}t)}}\,,$$

for all $s \in \mathbb{R}^m$.

By using a Feynman-Hellmann formula, we are led to introduce

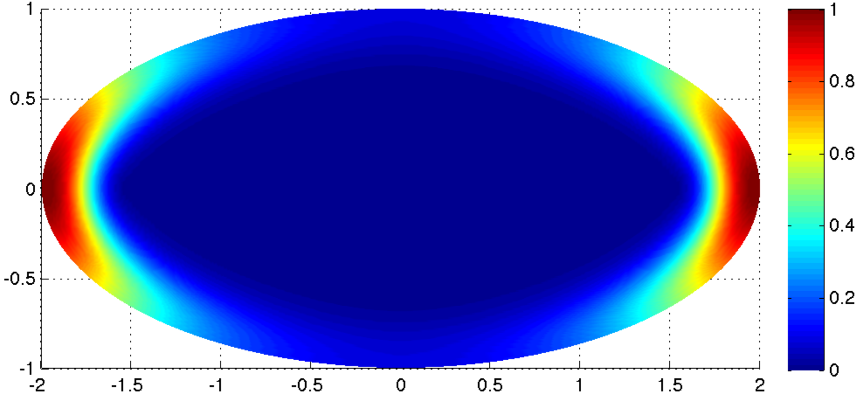
$$\mathsf{T} = \frac{1}{2}\left(\nabla_\xi \mu^\natural \cdot D_s + D_s \cdot \nabla_\xi \mu^\natural\right) ,$$

and we get the equation

$$\boxed{\mathsf{T} b_0 = \lambda_1 b_0} .$$

Then, λ_1 has to be chosen to solve the linearized transport equation at the singular point $s = 0$ and this condition is nothing but the belonging to the spectrum of the "harmonic oscillator" of symbol $\frac{1}{2}\mathrm{Hess}_{x_0,\xi_0}\mu$.

A fundamental application of our ideas, after **Fournais-Helffer**



(constant magnetic field and Neumann conditions)

We have recently made the following conjecture for the case of the magnetic ellipse:

$$\lambda_2(\hbar) - \lambda_1(\hbar) \underset{\hbar\to 0}{\sim}$$

$$\hbar^{\frac{13}{8}} \frac{2^{\frac{5}{2}}}{\sqrt{\pi}} \left(-\kappa''(0)\mu''(\zeta_0)\right)^{\frac{1}{4}} \left(\kappa(0) - \kappa\left(\frac{\ell}{4}\right)\right)^{\frac{1}{2}} C_1^{\frac{3}{4}} \mathsf{A} \left|\cos\left(\frac{\ell}{2}\left(\frac{\gamma_0}{h} - \frac{\zeta_0}{h^{\frac{1}{2}}} + \alpha_0\right)\right)\right| \mathrm{e}^{-\mathsf{S}/\hbar^{\frac{1}{4}}},$$

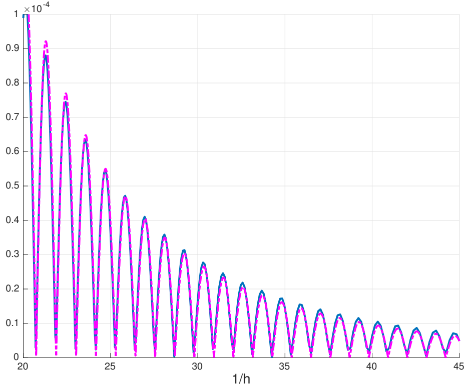
where

$$\mathsf{S} = \sqrt{\frac{2C_1}{\mu''(\zeta_0)}} \int_0^{\frac{\ell}{2}} \sqrt{\kappa(0) - \kappa(s)}\, \mathrm{d}s\,,$$

$$\mathsf{A} = \exp\left(-\int_{[0,\frac{\ell}{4}]} \frac{\partial_s \sqrt{\kappa(0) - \kappa(s)} - \sqrt{\frac{-\kappa''(0)}{2}}}{\sqrt{\kappa(0) - \kappa(s)}}\, \mathrm{d}s\right),$$

and where κ is the curvature of the boundary (maximal at 0 and π) and where all the constants are related to explicit model operators.

Numerically checked...



(simulations from 10 years ago versus our conjecture)

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- What is the relation between the Lorentz force and the magnetic Laplacian?
- Can we describe the magnetic bound states from the classical dynamics?

Preliminary comment

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Take $m = e = 1$. Consider an $\mathbf{A}$ such that $\mathbf{B} = \nabla \times \mathbf{A}$. The skew-symmetric matrix associated with $\mathbf{B}$ is

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This can be reformulated as

$$\boxed{\frac{d}{dt}\left(\dot{q} + \mathbf{A}(q)\right) = {}^t J_{\mathbf{A}}\dot{q}}\,.$$

By introducing the momentum variable $p = \dot{q} + \mathbf{A}(q)$, we see that (q, p) evolves according to the Hamiltonian flow associated with $H(q, p) = \frac{1}{2}\|p - \mathbf{A}(q)\|^2$.

For the rest of the talk, we are concerned with

$$\mathscr{L}_\hbar = (-i\hbar\nabla - \mathbf{A})^2$$

on $\mathbb{R}^d$, with $d = 2, 3$.
Its $\hbar$-symbol, in the Weyl quantization, is

$$H(q,p) = \|p - \mathbf{A}(q)\|^2 = \sum_{k=1}^{d}(p_k - A_k(q))^2.$$

The characteristic manifold of H is

$$\Sigma = \{(q,p) \in \mathbb{R}^{2d} : p = \mathbf{A}(q)\}.$$

The phase space $\mathbb{R}^d \times \mathbb{R}^d$ is equipped with the canonical symplectic form

$$\omega_0 = \mathrm{d}p \wedge \mathrm{d}q.$$

Did you say Weyl quantization?

We recall that

$$\mathrm{Op}_{\hbar}^{w}\, a\, \psi(q) = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^{2d}} e^{i\langle q-y,p\rangle/\hbar} a\left(\frac{q+y}{2}, p\right) \psi(y)\mathrm{d}y\mathrm{d}p\,,$$

for $\psi \in \mathcal{S}(\mathbb{R}^d)$. We have

$$\mathscr{L}_{\hbar} = \mathrm{Op}_{\hbar}^{w}\, H\,.$$

We introduce the parametrization of Σ

$$\mathbb{R}^d \ni q \mapsto j(q) := (q, \mathbf{A}(q)) \in \mathbb{R}^d \times \mathbb{R}^d$$

and its satisfies the magnetic-symplectic relation:

$$j^* \omega_0 = \mathrm{d}\alpha\,.$$

In the last years, Helffer and Kordyukov have intensively worked on the case of $\mathbb{R}^d$ with non-vanishing magnetic fields:

- In two dimensions, they have proved that $\lambda_n(\hbar)$ can be expanded in powers of $\hbar^{\frac{1}{2}}$.
- In three dimensions, via a tricky construction of quasimodes, they have conjectured that $\lambda_n(\hbar)$ could be expanded in powers of $\hbar^{\frac{1}{4}}$.

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- We have related the eigenvalues to the classical dynamics.
- Let us discuss two corollaries of our main normal form theorems.
- Many discussions with **F. Faure** and **Y. Colin de Verdière** have stimulated the proofs of these results.

In two dimensions

Theorem (**R.-Vũ Ngọc**, after **Helffer-Kordyukov**)

Let us assume that

(i) *B admits a unique minimum at q_0 that positive and non degenerate,*

(ii) $\liminf_{|q|\to+\infty} B(q) > b_0 := B(q_0)$.

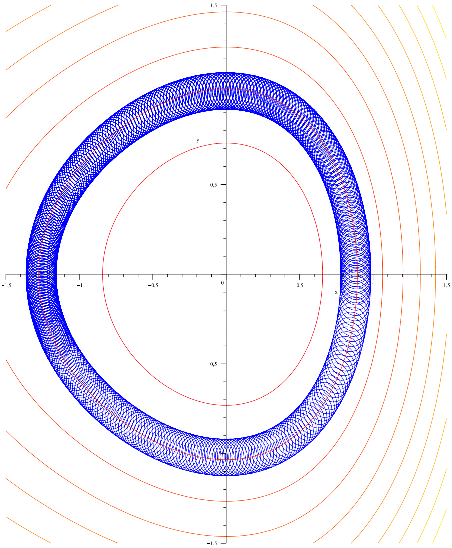
Then the eigenvalue $\lambda_m(\hbar)$ admit a full asymptotic expansion in $\hbar$ and

$$\lambda_m(\hbar) = b_0\hbar + \left[\theta^{2D}(q_0)\left(m - \frac{1}{2}\right) + \zeta^{2D}(q_0)\right]\hbar^2 + \mathcal{O}(\hbar^3)$$

where

$$\theta^{2D}(q_0) = \sqrt{\frac{\det \mathrm{Hess}_{q_0} B}{b_0^2}}\,.$$

This approximation is uniform in m as soon as m is of order $\hbar^{-1+\eta}$ for $\eta > 0$. The remainder becomes $\mathcal{O}(\hbar^{2+\eta})$.


1,5
0,5
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−1
−0,5
0
0,5
1
1,5
x
y

In three dimensions

Theorem (**Helffer-Kordyukov-R.-Vũ Ngọc**)

Let us assume that

(i) $b := \|\mathbf{B}\|$ *admits a unique minimum at* q_0 *that is positive and non degenerate,*

(ii) $\liminf_{|q|\to+\infty} b > b_0 := b(q_0)$ *and* $\|\nabla\mathbf{B}\| \leq C(1+\|\mathbf{B}\|)$.

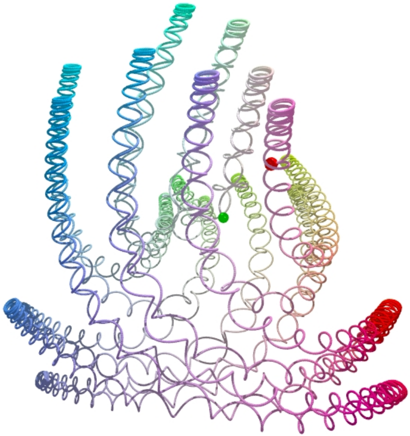
Then the m*-th eigenvalue admits a full asymptotic expansion in* $\hbar^{\frac{1}{2}}$ *and*

$$\lambda_m(\hbar) = b_0\hbar + \sigma^{3D}(q_0)\hbar^{\frac{3}{2}} + \left[\theta^{3D}(q_0)\left(m-\tfrac{1}{2}\right) + \zeta^{3D}(q_0)\right]\hbar^2 + \mathcal{O}(\hbar^{\frac{5}{2}})$$

where

$$\sigma^{3D}(q_0) = \sqrt{\frac{\mathrm{Hess}_{q_0} b\,(\mathbf{B},\mathbf{B})}{2b_0^2}}, \quad \theta^{3D}(q_0) = \sqrt{\frac{\det \mathrm{Hess}_{q_0} b}{\mathrm{Hess}_{q_0} b\,(\mathbf{B},\mathbf{B})}}.$$

This approximation is uniform in m as soon as m is of order $\hbar^{-\frac{1}{2}+\eta}$ for $\eta > 0$. The remainder becomes $\mathcal{O}(\hbar^{2+\eta})$.



General strategy

In order to obtain uniform estimates of the magnetic spectrum, we will:

- straighten the magnetic(-symplectic) geometry,
- implement a formal Birkhoff normal form,
- quantize the normal form via (adaptations of) the Egorov's theorem,
- establish (second) microlocal estimates (and establish some semiclassical Weyl estimates),
- repeat this procedure as often as necessary...

The manifold Σ is symplectic. Thus we may find local symplectic coordinates x_1, ξ_1, x_2, ξ_2 such that

(a) $z_1 = (x_1, \xi_1)$ represents the distance to Σ,

(b) z_2 parametrizes Σ.

In these coordinates, the Hamiltonian takes the form

$$H(z_1, z_2) = H^0 + \mathcal{O}(|z_1|^3), \quad \text{where} \quad H^0 = B(g^{-1}(z_2))|z_1|^2 .$$

Let us explain this in detail.

Normal symplectic coordinates

Lemma

For any $q \in \Omega$, the vectors

$$u_1 := \frac{1}{\sqrt{|B|}}(e_1, {}^t T_q \mathbf{A}(e_1)) , \quad v_1 := \frac{1}{\sqrt{|B|}}(e_2, {}^t T_q \mathbf{A}(e_2)) ,$$

form a symplectic basis of $T_{j(q)}\Sigma^\perp$.

Normal symplectic coordinates

Recall that

$$j^*\omega_0 = \mathrm{d}A \simeq B\,,$$

where $j : \mathbb{R}^2 \to \Sigma$ is the embedding $j(q) = (q, A(q))$.
There exists a diffeomorphism $g : \Omega \to g(\Omega) \subset \mathbb{R}^2_{z_2}$ such that $g(q_0) = 0$ and

$$g^*(\mathrm{d}\xi_2 \wedge \mathrm{d}x_2) = j^*\omega_0\,.$$

The new embedding $\tilde{j} := j \circ g^{-1} : \mathbb{R}^2 \to \Sigma$ is symplectic.

Normal symplectic coordinates

We introduce the map

$$\tilde{\Phi}(z_1, z_2) = \tilde{\jmath}(z_2) + x_1 u_1(\tilde{g}^{-1}(z_2)) + \xi_1 v_1(\tilde{g}^{-1}(z_2)) \,.$$

This map is not symplectic. The Jacobian matrix is symplectic for $z_1 = 0$. We can say that

$$\omega_0 - \tilde{\Phi}^* \omega_0$$

vanishes on $\{0\} \times \Omega$.

Normal symplectic coordinates

Lemma

Let us consider ω_0 and ω_1 two 2-forms on $\mathbb{R}^4$ which are closed and non-degenerate. Let us assume that $\omega_1 = \omega_0$ on $\{z_1 = 0\} \times \Omega$ where Ω is a bounded open set. In a neighborhood of $\{z_1 = 0\} \times \Omega$ there exists a change of coordinates ψ_1 such that

$$\psi_1^* \omega_1 = \omega_0 \quad \text{and} \quad \psi_1 = \mathsf{Id} + \mathscr{O}(|z_1|^2) .$$

Normal symplectic coordinates

Lemma

Let us consider ω_0 and ω_1 two 2-forms on $\mathbb{R}^4$ which are closed and non-degenerate. Let us assume that $\omega_1 = \omega_0$ on $\{z_1 = 0\} \times \Omega$ where Ω is a bounded open set. In a neighborhood of $\{z_1 = 0\} \times \Omega$ there exists a change of coordinates ψ_1 such that

$$\psi_1^* \omega_1 = \omega_0 \quad \textit{and} \quad \psi_1 = \mathsf{Id} + \mathscr{O}(|z_1|^2) \, .$$

The proof can be done with a **Moser** argument. See, for instance, **Hofer-Zehnder**, proof of Theorem 1.

Poincaré lemma

We can find a 1-form σ defined in a neighborhood of $z_1 = 0$ such that

$$\omega_1 - \omega_0 = \mathrm{d}\sigma \quad \text{and} \quad \sigma = \mathscr{O}(|z_1|^2)\,.$$

Moser argument

For $t \in [0,1]$, we let

$$\omega_t = \omega_0 + t(\omega_1 - \omega_0)\,.$$

The 2-form ω_t is closed and non-degenerate (up to choosing a neighborhood of $z_1 = 0$ small enough). We look for ψ_t such that

$$\psi_t^* \omega_t = \omega_0\,.$$

For that purpose, let us determine a vector field X_t such that

$$\frac{d}{dt}\psi_t = X_t(\psi_t)\,.$$

Moser argument

By using the Cartan formula, we get

$$0 = \frac{d}{dt}\psi_t^*\omega_t = \psi_t^*\left(\frac{d}{dt}\omega_t + \iota(X_t)\mathrm{d}\omega_t + \mathrm{d}(\iota(X_t)\omega_t)\right).$$

This becomes

$$\omega_0 - \omega_1 = \mathrm{d}(\iota(X_t)\omega_t),$$

and we are led to

$$\iota(X_t)\omega_t = -\sigma.$$

By non-degeneracy of ω_t, this determines X_t.

Moser argument

By using the Cartan formula, we get

$$0 = \frac{d}{dt}\psi_t^*\omega_t = \psi_t^*\left(\frac{d}{dt}\omega_t + \iota(X_t)\mathrm{d}\omega_t + \mathrm{d}(\iota(X_t)\omega_t)\right) .$$

This becomes

$$\omega_0 - \omega_1 = \mathrm{d}(\iota(X_t)\omega_t) ,$$

and we are led to

$$\iota(X_t)\omega_t = -\sigma .$$

By non-degeneracy of ω_t, this determines X_t.
Choosing a neighborhood of $(0,0)\times U$ small enough, we infer that ψ_t exists until the time $t=1$ and that it satisfies $\psi_t^*\omega_t = \omega_0$. Since $\sigma = \mathscr{O}(|z_1|^2)$, we get

$$\psi_1 = \mathsf{Id} + \mathscr{O}(|z_1|^2) .$$

Normal symplectic coordinates

We let $\Phi := \tilde{\Phi} \circ \psi_1$ and Φ is now symplectic. Elementary computations provide

$$\begin{aligned} H \circ \Phi(z_1, z_2) &= H \circ \Phi_{|z_1=0} + TH \circ \Phi_{|z_1=0}(z_1) + \frac{1}{2} T^2 (H \circ \Phi)_{|z_1=0}(z_1^2) + \mathscr{O}(|z_1|^3) \\ &= 0 + 0 + |B(g^{-1}(z_2))||z_1|^2 + \mathscr{O}(|z_1|^3)\,. \end{aligned}$$

We let $H^0 = |B(g^{-1}(z_2))||z_1|^2$.

Proposition

For $\gamma \in \mathcal{O}_3$, there exist two formal series $\tau, \kappa \in \mathcal{O}_3$ such that

$$e^{i\hbar^{-1}\,\mathrm{ad}_\tau}(H^0 + \gamma) = H^0 + \kappa\,,$$

with $[\kappa, |z_1|^2] = 0$.

Explicitly,

$$[\kappa_1, \kappa_2](x, \xi, \hbar) = 2\sinh\left(\frac{\hbar}{2i}\Box\right)\left(\kappa_1(t, \tau, \hbar)\kappa_2(y, \eta, \hbar)\right)\Big|_{\substack{t=y=x,\\ \tau=\eta=\xi}}$$

where

$$\Box = \sum_{j=1}^{2} \partial_{\tau_j}\partial_{y_j} - \partial_{t_j}\partial_{\eta_j}\,.$$

Theorem (R.-Vũ Ngọc, after Ivrii)

For h small enough, the exists a Fourier Integral Operator $U_\hbar$ such that

$$U_\hbar^* U_\hbar = I + Z_\hbar, \qquad U_\hbar U_\hbar^* = I + Z_\hbar',$$

where $Z_\hbar, Z_\hbar'$ are pseudors vanishing microlocally in a neighborhood of $\tilde{\Omega} \cap \Sigma$ and such that

$$U_\hbar^* \mathcal{L}_{\hbar,\mathbf{A}} U_\hbar = \mathcal{N}_\hbar + R_\hbar,$$

- *$\mathcal{N}_\hbar$ is a pseudor belonging to $S(m)$ and commuting with*
$$\mathcal{I}_\hbar := -\hbar^2 \frac{\partial^2}{\partial x_1^2} + x_1^2,$$
- *$\mathcal{N}_\hbar = \mathcal{H}_\hbar^0 + Q_\hbar$, where $\mathcal{H}_\hbar^0 = \mathrm{Op}_\hbar^w(H^0)$, $H^0 = B(\varphi^{-1}(z_2))|z_1|^2$, and $Q_\hbar$ commutes with $\mathcal{I}_\hbar$ and is relatively bounded with respect to $\mathcal{H}_\hbar^0$ with arbitrary small bound.*

We assume that the magnetic field does not vanish and is confining:

$$\exists \tilde{C}_1 > 0, \quad M_0 > 0, \quad B(q) \geq \tilde{C}_1 \quad \text{for} \quad |q| \geq M_0.$$

Theorem

Let $0 < C_1 < \tilde{C}_1$. Then, the spectra of $\mathcal{L}_\hbar$ and $\mathcal{N}_\hbar = \mathcal{H}^0_\hbar + Q_\hbar$ in $(-\infty, C_1\hbar]$ are discrete. Let $0 < \lambda_1(\hbar) \leq \lambda_2(\hbar) \leq \cdots$ be the eigenvalues of $\mathcal{L}_\hbar$ and $0 < \mu_1(\hbar) \leq \mu_2(\hbar) \leq \cdots$ the one of $\mathcal{N}_\hbar$. Then, for all $j \in \mathbb{N}^*$ such that $\lambda_j(\hbar) \leq C_1\hbar$ and $\mu_j(\hbar) \leq C_1\hbar$, we have

$$|\lambda_j(\hbar) - \mu_j(\hbar)| = \mathcal{O}(\hbar^\infty).$$

Now, I present the result obtained with **B. Helffer**, **Y. Kordyukov** and **Vũ Ngọc** in three dimensions.

Moving magnetic frame

Let us assume that $\mathbf{B}(0) \neq 0$ so that $\mathbf{B}$ is not 0 near 0. We can even assume that $\mathbf{B}(0) = \|\mathbf{B}(0)\|\mathbf{e}_3$ and we define

$$\mathbf{b} = \frac{\mathbf{B}}{\|\mathbf{B}\|}$$

and the smooth vectors $\mathbf{c}$ and $\mathbf{d}$ so that $(\mathbf{b}, \mathbf{c}, \mathbf{d})$ is a direct orthonormal basis.

Straightening the magnetic 2-form

We introduce the coordinate along the magnetic field, via:

$$\partial_3 \chi(\hat{q}) = \mathbf{b}(\chi(\hat{q}))\,, \quad \chi^*(\mathrm{d}\alpha) = \mathrm{d}\hat{q}_1 \wedge \mathrm{d}\hat{q}_2\,.$$

Note that

(i) $\mathbf{b}$ belongs to the kernel of the magnetic 2-form $\mathrm{d}\alpha$.

(ii) $j^*\omega_0 = \mathrm{d}\alpha$.

Basis of the tangent space

We can reparametrize Σ

$$\begin{aligned} \iota : \hat{\Omega} &\longrightarrow \Sigma \\ \hat{q} &\mapsto (\chi(\hat{q}), A_1(\chi(\hat{q}))\,, A_2(\chi(\hat{q})), A_3(\chi(\hat{q})))\,, \end{aligned}$$

and define a basis of the tangent space $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$:

$$\mathbf{f}_j = (T\chi(\mathbf{e}_j), T\mathbf{A} \circ T\chi(\mathbf{e}_j)),\ j = 1, 2, 3\,.$$

We notice that

$$\omega_0(\mathbf{f}_j, \mathbf{f}_k) = \mathrm{d}\alpha(T\chi(\mathbf{e}_j), T\chi(\mathbf{e}_k)) = \chi^* \mathrm{d}\alpha(\mathbf{e}_j, \mathbf{e}_k) = \mathrm{d}\hat{q}_1 \wedge \mathrm{d}\hat{q}_2(\mathbf{e}_j, \mathbf{e}_k)\,.$$

Basis of the symplectic orthogonal

The following vectors of $\mathbb{R}^3 \times \mathbb{R}^3$ form a basis of the symplectic orthogonal of $T_{\iota(\hat{q})}\Sigma$:

$$\mathbf{f}_4 = \|\mathbf{B}\|^{-1/2}(\mathbf{c}, ({}^t T_{\chi(\hat{q})}\mathbf{A})\mathbf{c}), \quad \mathbf{f}_5 = \|\mathbf{B}\|^{-1/2}(\mathbf{d}, ({}^t T_{\chi(\hat{q})}\mathbf{A})\mathbf{d}).$$

We need a sixth vector. We introduce $\mathbf{f}_6 = (0, \mathbf{b}) + \rho_1 \mathbf{f}_1 + \rho_2 \mathbf{f}_2$ where ρ_1 and ρ_2 are determined by the relations $\omega_0(\mathbf{f}_j, \mathbf{f}_6) = 0$ for $j = 1, 2$.

Lemma

The family $(\mathbf{f}_j)_{j=1,\dots,6}$ is a symplectic basis.

We introduce the local diffeomorphism

$$(x, \xi) \mapsto \iota(x_2, \xi_2, x_3) + x_1 \mathbf{f}_4(x_2, \xi_2, x_3) + \xi_1 \mathbf{f}_5(x_2, \xi_2, x_3) + \xi_3 \mathbf{f}_6(x_2, \xi_2, x_3)\,.$$

It is symplectic "on" Σ. Thus we can make it symplectic modulo a correction that is tangent to the identity (Moser-Weinstein's lemma). In these new coordinates, H becomes

$$\xi_3^2 + b(x_2, \xi_2, x_3)(x_1^2 + \xi_1^2) + \mathcal{O}(|x_1|^3 + |\xi_1|^3 + |\xi_3|^3)\,.$$

Formal series of pseudo-differential operators

We consider the space $\mathcal{E}$ of formal series in $(x_1, \xi_1, \xi_3, \hbar)$ with smooth coefficients in (x_2, ξ_2, x_3) :

$$\mathcal{E} = \mathscr{C}^{\infty}_{x_2,\xi_2,x_3}[[x_1, \xi_1, \xi_3, \hbar]] .$$

We equip $\mathcal{E}$ of the semiclassical Moyal product $\star$ (w.r.t. all the variables) and the commutator of κ_1 and κ_2 is by definition

$$[\kappa_1, \kappa_2] = \kappa_1 \star \kappa_2 - \kappa_2 \star \kappa_1 .$$

The degree of $x_1^{\alpha_1}\xi_1^{\alpha_2}\xi_3^{\beta}\hbar^{\ell}$ is $\alpha_1 + \alpha_2 + \beta + 2\ell = |\alpha| + \beta + 2\ell$. The space of formal series with valuation at least N is denoted by $\mathcal{O}_N$. For all $\tau, \gamma \in \mathcal{E}$, we let $\mathrm{ad}_\tau\, \gamma = [\tau, \gamma]$.

Proposition

For $\gamma \in \mathcal{O}_3$, there exist two formal series $\tau, \kappa \in \mathcal{O}_3$ such that

$$\boxed{e^{i\hbar^{-1}\,\mathrm{ad}_\tau}(H^0 + \gamma) = H^0 + \kappa},$$

with $[\kappa, |z_1|^2] = 0$ and $H^0 = \xi_3^2 + b(x_2, \xi_2, x_3)|z_1|^2$.

We may write κ in the form

$$\kappa = \sum_{k\geq 3} \sum_{2\ell+2m+\beta=k} \hbar^{\ell} c_{\ell,m}(x_2,\xi_2,x_3)|z_1|^{2m}\xi_3^{\beta}.$$

This series may be rearranged:

$$\kappa = \sum_{k\geq 3} \sum_{2\ell+2m+\beta=k} \hbar^{\ell} c^{\star}_{\ell,m}(x_2,\xi_2,x_3)(|z_1|^{2})^{\star m}\xi_3^{\beta}.$$

Theorem (First normal form, after Ivrii)

If $\mathbf{B}(q_0) \neq 0$, there exists a neighborhood $\mathcal{U}_0$ of $(q_0, \mathbf{A}(q_0))$ and symplectic coordinates $(x_1, \xi_1, x_2, \xi_2, x_3, \xi_3)$ such that $\Sigma = \{x_1 = \xi_1 = \xi_3 = 0\}$ and a FIO $U_\hbar$ microlocally unitary near $\mathcal{U}_0$ and a smooth function, with compact support in Z and ξ_3, $f^\star(\hbar, Z, x_2, \xi_2, x_3, \xi_3)$ whose Taylor expansion $Z, \xi_3, \hbar$ is

$$\sum_{k\geq 3} \sum_{2\ell+2m+\beta=k} \hbar^\ell c^\star_{\ell,m}(x_2, \xi_2, x_3) Z^m \xi_3^\beta$$

so that

$$U_\hbar^* \mathcal{L}_\hbar U_\hbar = \mathcal{N}_\hbar + \mathcal{R}_\hbar\,,$$

with

$$\mathcal{N}_\hbar = \hbar^2 D_{x_3}^2 + \mathcal{I}_\hbar \operatorname{Op}_\hbar^w b + \operatorname{Op}_\hbar^w f^\star(\hbar, \mathcal{I}_\hbar, x_2, \xi_2, x_3, \xi_3)\,,$$

Confinement

Assumption

We assume that

$$b(q) \geq b_0 := \inf_{q\in\mathbb{R}^3} b(q) > 0\,,$$

and that there exists $C > 0$ such that

$$\|\nabla \mathbf{B}(q)\| \leq C\,(1 + b(q))\,,\ \forall q \in \mathbb{R}^3\,.$$

By the Persson theorem and an Helffer-Morame theorem, the bottom of the essential spectrum is asymptotically larger than $\hbar b_1$, where

$$b_1 := \liminf_{|q|\to+\infty} b(q)$$

Confinement

Assumption

We assume that

$$0 < b_0 < b_1 \,.$$

This assumption ensures that the infimum of b *is attained (say at* 0 *with* $\mathbf{A}(0) = 0$*). We also assume that (confinement near* 0*) there exist* $\varepsilon > 0$ *and* $\beta_0 \in (b_0, b_1)$ *such that*

$$\{b(q) \leq \beta_0\} \subset D(0, \varepsilon)\,.$$

Corollary

We introduce

$$\mathcal{N}_{\hbar}^{\sharp} = \mathrm{Op}_{\hbar}^{w}\left(N_{\hbar}^{\sharp}\right),$$

with

$$N_{\hbar}^{\sharp} = \xi_3^2 + \mathcal{I}_{\hbar}\underline{b}(x_2, \xi_2, x_3) + f^{\star,\sharp}(\hbar, \mathcal{I}_{\hbar}, x_2, \xi_2, x_3, \xi_3)$$

and where $\underline{b}$ *is a convenient extension of* b *outside* $D(0, \varepsilon)$ *and where* $f^{\star,\sharp} = \chi(x_2, \xi_2, x_3) f^{\star}$*, with* χ *is a smooth cutoff being* 1 *near* $D(0, \varepsilon)$*. We also introduce*

$$\mathcal{N}_{\hbar}^{[1],\sharp} = \mathrm{Op}_{\hbar}^{w}\left(N_{\hbar}^{[1],\sharp}\right),$$

where $N_{\hbar}^{[1],\sharp} = \xi_3^2 + \hbar\underline{b}(x_2, \xi_2, x_3) + f^{\star,\sharp}(\hbar, \hbar, x_2, \xi_2, x_3, \xi_3)$.

Corollary (continued)

If ε and the support of $f^{\star,\sharp}$ are small enough, then

(a) *The spectra of $\mathcal{L}_\hbar$ and $\mathcal{N}_\hbar^\sharp$ sous $\beta_0\hbar$ coincide modulo $\mathcal{O}(\hbar^\infty)$.*

(b) *For all $c \in (0, \min(3b_0, \beta_0))$, the spectra $\mathcal{L}_\hbar$ and $\mathcal{N}_\hbar^{[1],\sharp}$ under $c\hbar$ coincide modulo $\mathcal{O}(\hbar^\infty)$.*

Assumption

We assume that b admits a unique minimum at 0 (that is positive) and that $T_0^2 b(\mathbf{B}(0), \mathbf{B}(0)) > 0$.

We have $\partial_3 b(0,0,0) = 0$ and in the coordinates (x_2, ξ_2, x_3),

$$\partial_3^2 b(0,0,0) > 0\,.$$

It follows from the implicit functions theorem that, for x_2 small enough, there exists a smooth function $(x_2, \xi_2) \mapsto s(x_2, \xi_2)$, $s(0,0) = 0$, such that

$$\partial_3 b(x_2, \xi_2, s(x_2, \xi_2)) = 0\,.$$

We let

$$\nu(x_2, \xi_2) := \left(\tfrac{1}{2}\partial_3^2 b(x_2, \xi_2, s(x_2, \xi_2))\right)^{1/4}\,.$$

Theorem

There exist a neighborhood $\mathcal{V}_0$ of 0 and a FIO $V_\hbar$ microlocally unitary near $\mathcal{V}_0$ and such that

$$V_\hbar^* \mathcal{N}_\hbar^{[1]} V_\hbar =: \underline{\mathcal{N}}_\hbar^{[1]} = \mathrm{Op}_\hbar^w\left(\underline{N}_\hbar^{[1]}\right),$$

where $\underline{N}_\hbar^{[1]} = \nu^2(x_2,\xi_2)\left(\xi_3^2 + \hbar x_3^2\right) + \hbar b(x_2,\xi_2,s(x_2,\xi_2)) + \underline{R}_\hbar$ *and* $\underline{R}_\hbar$ *is a semiclassical symbol* $\underline{R}_\hbar = \mathcal{O}(\hbar x_3^3) + \mathcal{O}(\hbar\xi_3^2) + \mathcal{O}(\xi_3^3) + \mathcal{O}(\hbar^2)$.

Corollary

We introduce

$$\underline{\mathcal{N}}_{\hbar}^{[1],\sharp} = \mathrm{Op}_{\hbar}^{w}\left(\underline{N}_{\hbar}^{[1],\sharp}\right),$$

where $\underline{N}_{\hbar}^{[1],\sharp} = \underline{\nu}^2(x_2,\xi_2)\left(\xi_3^2 + \hbar x_3^2\right) + \hbar \underline{b}(x_2,\xi_2,s(x_2,\xi_2)) + \underline{R}_{\hbar}^{\sharp}$, *with* $\underline{R}_{\hbar}^{\sharp} = \chi(x_2,\xi_2,x_3,\xi_3)\underline{R}_{\hbar}$, *and where* $\underline{\nu}$ *is a convenient extension of* ν.
If ε *and the support of* χ *are small enough, then*

(a) *The spectra of* $\underline{\mathcal{N}}_{\hbar}^{[1],\sharp}$ *and* $\mathcal{N}_{\hbar}^{[1],\sharp}$ *below* $\beta_0\hbar$ *coincide modulo* $\mathcal{O}(\hbar^{\infty})$.

(b) *For all* $c \in (0, \min(3b_0, \beta_0))$, *the spectra of* $\mathcal{L}_{\hbar}$ *and* $\underline{\mathcal{N}}_{\hbar}^{[1],\sharp}$ *below* $c\hbar$ *coincide modulo* $\mathcal{O}(\hbar^{\infty})$.

Towards the second microlocalization

We let $h = \hbar^{\frac{1}{2}}$ and, if $A_\hbar$ is a semiclassical symbol on $T^*\mathbb{R}^2$, having an expansion in $\hbar^{\frac{1}{2}}$, we write

$$\mathcal{A}_\hbar := \mathrm{Op}_\hbar^w A_\hbar = \mathrm{Op}_h^w \mathsf{A}_h =: \mathfrak{A}_h \,,$$

with

$$\mathsf{A}_h(x_2, \tilde{\xi}_2, x_3, \tilde{\xi}_3) = A_{h^2}(x_2, h\tilde{\xi}_2, x_3, h\tilde{\xi}_3) \,.$$

Theorem

There exist a unitary operator W_h and a smooth function $g^\star$, with arbitrarily small compact support, with respect to the second variable Z and compactly supported in (x_2, ξ_2) such that the Taylor series in Z, h is

$$\sum_{2m+2\ell\geq 3} c_{m,\ell}(x_2,\xi_2) Z^m h^\ell\,,$$

so that

$$W_h^* \underline{\mathfrak{N}}_h^{[1],\sharp} W_h =: \mathfrak{M}_h = \mathrm{Op}_h^w\left(\mathsf{M}_h\right)\,,$$

with

$$\mathsf{M}_h = h^2 \underline{b}(x_2, h\tilde{\xi}_2, s(x_2, h\tilde{\xi}_2)) + h^2 \mathcal{J}_h\, \mathrm{Op}_h^w\, \underline{\nu}^2(x_2, h\tilde{\xi}_2) + h^2 g^\star(h, \mathcal{J}_h, x_2, h\tilde{\xi}_2) + h^2 \mathsf{R}_h\,.$$

Theorem (continued)

where

(a) *the operator* $\underline{\mathfrak{N}}_{h}^{[1],\sharp}$ *is* $\underline{\mathcal{N}}_{\hbar}^{[1],\sharp}$,

(b) *we have let* $\mathcal{J}_h = \tilde{\xi}_3^2 + x_3^2$,

(c) *the remainder* R_h *satisfies* $\mathrm{R}_h(x_2, h\tilde{\xi}_2, x_3, \tilde{\xi}_3) = \mathcal{O}((x_3, \tilde{\xi}_3)^\infty)$.

Corollary

We have

$$\mathfrak{M}_h^\sharp = \mathrm{Op}_h^w\left(\mathsf{M}_h^\sharp\right),$$

with

$$\mathsf{M}_h^\sharp = h^2\underline{b}(x_2, h\tilde{\xi}_2, s(x_2, h\tilde{\xi}_2)) + h^2\mathcal{J}_h\underline{\nu}^2(x_2, h\tilde{\xi}_2) + h^2 g^\star(h, \mathcal{J}_h, x_2, h\tilde{\xi}_2)\,.$$

We define

$$\mathfrak{M}_h^{[1],\sharp} = \mathrm{Op}_h^w\left(\mathsf{M}_h^{[1],\sharp}\right),$$

where

$$\mathsf{M}_h^{[1],\sharp} = h^2\underline{b}(x_2, h\tilde{\xi}_2, s(x_2, h\tilde{\xi}_2)) + h^3\underline{\nu}^2(x_2, h\tilde{\xi}_2) + h^2 g^\star(h, h, x_2, h\tilde{\xi}_2)\,.$$

Corollary

If ε and the support of $g^\star$ are small enough, we have:

(a) *For all $\eta > 0$, the spectra of $\underline{\mathfrak{N}}_h^{[1],\sharp}$ and $\mathfrak{M}_h^\sharp$ below $b_0 h^2 + \mathcal{O}(h^{2+\eta})$ coincide modulo $\mathcal{O}(h^\infty)$.*

(b) *For $c \in (0,3)$, the spectra of $\mathfrak{M}_h^\sharp$ and $\mathfrak{M}_h^{[1],\sharp}$ below $b_0 h^2 + c\nu^2(0,0)h^3$ coincide modulo $\mathcal{O}(h^\infty)$.*

(c) *If $c \in (0,3)$, the spectra of $\mathcal{L}_\hbar$ and $\mathcal{M}_\hbar^{[1],\sharp}$ below $b_0\hbar + c\nu^2(0,0)\hbar^{\frac{3}{2}}$ coincide modulo $\mathcal{O}(\hbar^\infty)$.*

Formal series

We define an appropriate space of formal series in $(x_3, \tilde{\xi}_3, h)$. Let us consider

$$\mathcal{F} := \{d \text{ s. t. } \exists c \in S^0(\mathbb{R}^4) : d(x_2, \tilde{\xi}_2; \mu, h) = c(x_2, \mu\tilde{\xi}_2; \mu, h)\},$$

and

$$\mathcal{E} := \mathcal{F}[[x_3, \tilde{\xi}_3, h]],$$

equipped with the Poisson bracket

$$\mathcal{E}^2 \ni (f, g) \mapsto \{f, g\} = \sum_{j=2,3} \frac{\partial f}{\partial \tilde{\xi}_j}\frac{\partial g}{\partial x_j} - \frac{\partial g}{\partial \tilde{\xi}_j}\frac{\partial f}{\partial x_j} \in \mathcal{E},$$

and of the Moyal product $[f, g]$.

Assumption

The function b admits a unique minimum en 0 (positive) and non degenerate.

Theorem

There exist a $\hbar$-FIO unitary $Q_{\hbar^{\frac{1}{2}}}$ whose phase may be expanded in powers ofe $\hbar^{\frac{1}{2}}$ and a smooth function $k^\star$, with compact support in Z, such that

$$Q^*_{\hbar^{\frac{1}{2}}}\mathcal{M}^{[1],\sharp}_{\hbar}Q_{\hbar^{\frac{1}{2}}}=\mathcal{F}_\hbar+\mathcal{G}_\hbar\,,$$

where

(a) $\mathcal{F}_\hbar$ *is the operator* $b_0\hbar+\nu^2(0,0)\hbar^{\frac{3}{2}}-\frac{\|(\nabla\nu^2)(0,0)\|^2}{2\theta}\hbar^2+\hbar\left(\frac{\theta}{2}\mathcal{K}_\hbar+k^\star(\hbar^{\frac{1}{2}},\mathcal{K}_\hbar)\right)$,

(b) *the function* $k^\star$ *satisfies* $k^\star(\hbar^{\frac{1}{2}},Z)=\mathcal{O}((\hbar^{\frac{1}{2}},Z^{\frac{1}{2}})^3)$,

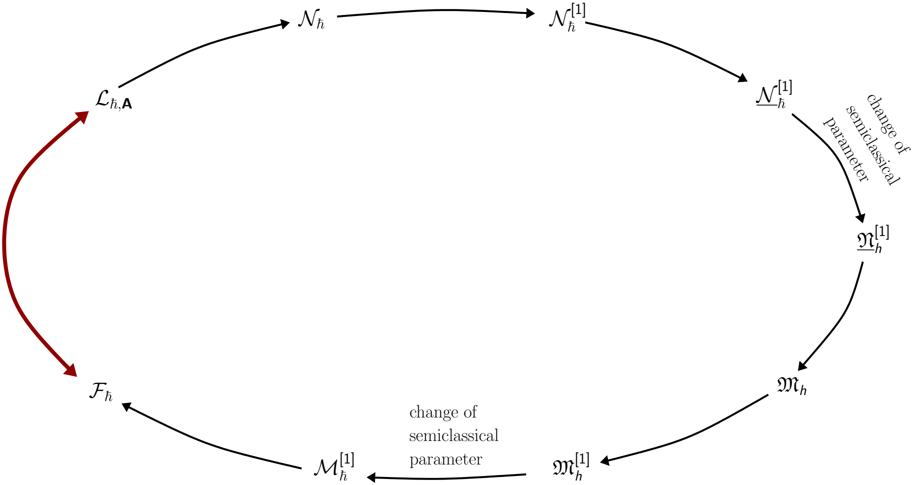
(c) *the remainder is in the form* $\mathcal{G}_\hbar=\mathrm{Op}^w_\hbar(G_\hbar)$, *with* $G_\hbar=\hbar\mathcal{O}(|z_2|^\infty)$.

Corollary

If ε and the support of k are small enough, we have

(a) *For all $\eta \in \left(0, \frac{1}{2}\right)$, the spectra of $\mathcal{M}_{\hbar}^{[1],\sharp}$ and $\mathcal{F}_{\hbar}$ below $b_0\hbar + \mathcal{O}(\hbar^{1+\eta})$ coincide modulo $\mathcal{O}(\hbar^{\infty})$.*

(b) *For all $c \in (0,3)$, the spectra of $\mathcal{L}_{\hbar}$ and $\mathcal{F}_{\hbar}$ below $b_0\hbar + c\nu^2(0,0)\hbar^{\frac{3}{2}}$ coincide modulo $\mathcal{O}(\hbar^{\infty})$.*


$\mathcal{N}_\hbar$
$\mathcal{N}_\hbar^{[1]}$
$\underline{\mathcal{N}}_\hbar^{[1]}$
change of semiclassical parameter
$\underline{\mathfrak{N}}_h^{[1]}$
$\mathfrak{M}_h$
$\mathfrak{M}_h^{[1]}$
change of semiclassical parameter
$\mathcal{M}_\hbar^{[1]}$
$\mathcal{F}_\hbar$
$\mathcal{L}_{\hbar,\mathbf{A}}$

We consider $\Omega \subset \mathbb{R}^2$, bounded and $\mathscr{L}_\hbar = (-i\hbar\nabla - \mathbf{A})^2$. We assume B is analytic in a neighborhood of Ω.

Assumption

$B_{|\overline{\Omega}}$ has a non-degenerate local and positive minimum at $(0,0)$. Moreover, we can write

$$B(x_1, x_2) = b_0 + \alpha x_1^2 + \gamma x_2^2 + \mathscr{O}(\|x\|^3)\,, \qquad \textit{with } 0 < \alpha \leq \gamma\,.$$

Theorem (Bonthonneau-R.)

Let $\ell \in \mathbb{N}$. There exist

i. *a neighborhood $\mathcal{V} \subset \Omega$ of $(0,0)$,*

ii. *an analytic function S on $\mathcal{V}$ satisfying*

$$\operatorname{Re} S(x) = \frac{b_0}{2}\left[\frac{\sqrt{\alpha}}{\sqrt{\alpha}+\sqrt{\gamma}}x_1^2 + \frac{\sqrt{\gamma}}{\sqrt{\alpha}+\sqrt{\gamma}}x_2^2\right] + \mathscr{O}(\|x\|^3)\,,$$

iii. *a sequence of analytic functions $(a_j)_{j\in\mathbb{N}}$ on $\mathcal{V}$,*

iv. *a sequence of real numbers $(\mu_j)_{j\in\mathbb{N}}$ satisfying*

$$\mu_0 = b_0\,, \quad \mu_1 = 2\ell\frac{\sqrt{\alpha\gamma}}{b_0} + \frac{(\sqrt{\alpha}+\sqrt{\gamma})^2}{2b_0}\,,$$

such that, for all $J \in \mathbb{N}$, and uniformly in $\mathcal{V}$,

$$e^{S/\hbar}\left((-i\hbar\nabla - \mathbf{A})^2 - \hbar\sum_{j\geq 0}^{J}\mu_j\hbar^j\right)\left(e^{-S/\hbar}\sum_{j\geq 0}^{J}a_j\hbar^j\right) = \mathscr{O}(\hbar^{J+2})\,.$$

Preliminary comment: WKB analysis and normal form

There exist a Fourier Integral Operator $U_\hbar$, quantizing a canonical transformation, and a smooth function $f_\hbar$ such that, locally in space near 0 and microlocally near the characteristic manifold of $\mathscr{L}_\hbar$,

$$\boxed{U_\hbar{}^* \mathscr{L}_\hbar U_\hbar = \mathrm{Op}_h^{\mathrm{w}} f_\hbar(\mathcal{H}, z_2) + \mathscr{O}(\hbar^\infty)}\,.$$

where $\mathcal{H} = h^2 D_{x_1}^2 + x_1^2$. Moreover, $f_\hbar(Z, z_2) = Z\hat{B}(z_2) + \mathscr{O}(\hbar^2) + \mathscr{O}(Z^2)$, where $\hat{B}$ is the magnetic field "seen" on the characteristic manifold.

Preliminary comment: WKB analysis and normal form

If we are interested in the low lying eigenvalues (which are essentially in the form $b_0\hbar + \mu_1\hbar^2$), we can look for a L^2-normalized WKB Ansatz expressed in normal coordinates as

$$\boxed{\Psi_\hbar(x_1, x_2) = g_\hbar(x_1)\psi_\hbar(x_2)}\,,$$

where $g_\hbar$ is the first normalized eigenfunction of $\mathcal{H}$. We find the effective eigenvalue equation

$$\mathrm{Op}_\hbar^{\mathrm{w}}(\hat{B} - b_0)\psi_\hbar = \mu_1\hbar\psi_\hbar + \mathscr{O}(\hbar^2)\,,$$

in which we insert the Ansatz $\psi_\hbar = e^{-S/\hbar}a$. We get

$$\boxed{\hat{B}(x_2, -iS'(x_2)) = b_0}\,.$$

We consider the conjugated operator acting locally as

$$\mathscr{L}_\hbar^S = e^{S/\hbar}\mathscr{L}_\hbar e^{-S/\hbar} = (\hbar D_{x_1} - A_1 + i\partial_{x_1} S)^2 + (\hbar D_{x_2} - A_2 + i\partial_{x_2} S)^2 \,.$$

We have

$$\mathscr{L}_\hbar^S = (-A_1 + i\partial_{x_1} S)^2 + (-A_2 + i\partial_{x_2} S)^2 + i\hbar\nabla\cdot\mathbf{A} - \hbar^2\Delta + \hbar\Delta S + 2\hbar(\nabla S + i\mathbf{A})\cdot\nabla \,.$$

We seek to determine S so that there exist a family of functions $(a_j)_{j\in\mathbb{N}}$ defined in a neighborhood of $(0,0)$ and a sequence of real numbers $(\mu_j)_{j\in\mathbb{N}}$ such that, in the sense of asymptotic series,

$$\boxed{\mathscr{L}_\hbar^S\left(\sum_{j\geq 0}\hbar^j a_j\right) \sim \hbar\left(\sum_{j\geq 0}\mu_j\hbar^j\right)\left(\sum_{j\geq 0}\hbar^j a_j\right)}\,.$$

Choice of gauge

Lemma

There exists an analytic and real-valued function φ, in a neighborhood of Ω, such that

$$\Delta\varphi = B\,,\quad \varphi(x_1,x_2) = \frac{B(0,0)}{4}(x_1^2+x_2^2) + \mathscr{O}(\|x\|^3)\,.$$

We let

$$\mathbf{A} = (\nabla\varphi)^\perp\,.$$

An effective eikonal equation

If $a : \mathbb{R}^2 \to \mathbb{C}$ is an analytic function near $(0,0) \in \mathbb{R}^2$, one denotes by $\tilde{a}$ the function defined near $(0,0) \in \mathbb{C}^2$ by

$$\tilde{a}(z,w) = a\left(\frac{z+w}{2}, \frac{z-w}{2i}\right) .$$

We have $\tilde{a}(z,\overline{z}) = a(\mathrm{Re}\, z, \Im z)$.

Lemma

There exists a holomorphic function w defined in a neighborhood of 0 satisfying

$$\boxed{\tilde{B}(z, w(z)) = b_0} .$$

and such that

$$w(0) = 0 , \quad w'(0) = \frac{\sqrt{\gamma} - \sqrt{\alpha}}{\sqrt{\gamma} + \sqrt{\alpha}} .$$

Lemma

Consider a function w given by the previous lemma and, in a neighborhood of 0, the holomorphic function defined by

$$\boxed{f(z) = -2\int_{[0,z]} \partial_z \tilde{\varphi}(\zeta, w(\zeta))\mathrm{d}\zeta} \,.$$

We have

$$f(0) = 0\,, \quad f'(0) = 0\,, \quad f''(0) = \frac{b_0}{2}\frac{\sqrt{\alpha}-\sqrt{\gamma}}{\sqrt{\gamma}+\sqrt{\alpha}}\,.$$

In particular, letting $S = \varphi + f$, we have

$$\mathrm{Re}\, S(x) = \frac{b_0}{2}\left[\frac{\sqrt{\alpha}}{\sqrt{\alpha}+\sqrt{\gamma}}x_1^2 + \frac{\sqrt{\gamma}}{\sqrt{\alpha}+\sqrt{\gamma}}x_2^2\right] + \mathscr{O}(\|x\|^3)\,.$$

Determining the phase modulo holomorphic functions

Collecting the terms of order 0, we get

$$(-A_1 + i\partial_{x_1} S)^2 + (-A_2 + i\partial_{x_2} S)^2 = 0 \,,$$

and thus

$$(-A_1 + i\partial_{x_1} S + i(-A_2 + i\partial_{x_2} S))(-A_1 + i\partial_{x_1} S - i(-A_2 + i\partial_{x_2} S)) = 0 \,.$$

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Let us consider an S such that

$$-A_1 + i\partial_{x_1} S + i(-A_2 + i\partial_{x_2} S) = 0\,,$$

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Let us consider an S such that

$$-A_1 + i\partial_{x_1} S + i(-A_2 + i\partial_{x_2} S) = 0\,,$$

so that

$$\boxed{2\partial_{\overline{z}} S = -iA_1 + A_2}\,, \qquad \partial_{\overline{z}} = \frac{1}{2}\left(\partial_{x_1} + i\partial_{x_2}\right)\,.$$

Determining the phase modulo holomorphic functions

Collecting the terms of order 0, we get

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so that

$$2\partial_{\overline{z}} S = -iA_1 + A_2 \,, \qquad \partial_{\overline{z}} = \frac{1}{2}\left(\partial_{x_1} + i\partial_{x_2}\right) \,.$$

We have $2\partial_{\overline{z}}\varphi = -iA_1 + A_2$ and thus S is in the form

$$S = \varphi + f(z) \,,$$

where f is a holomorphic function near $(0,0)$.

Rewriting the operator in terms of complex derivatives

Note that $\Delta = 4\partial_z\partial_{\overline{z}}$ and thus

$$\Delta S = B\,.$$

We get

$$\mathscr{L}_\hbar^S = -\hbar^2\Delta + \hbar B + 2\hbar(\nabla S + i\mathbf{A})\cdot\nabla\,.$$

We have

$$(\nabla S + i\mathbf{A})\cdot\nabla = (\partial_1 S + iA_1)\partial_1 + (\partial_2 S + iA_2)\partial_2$$

so that

$$(\nabla S + i\mathbf{A})\cdot\nabla = (\partial_1\varphi - i\partial_2\varphi + f'(z))\partial_1 + (\partial_2\varphi + i\partial_1\varphi + if'(z))\partial_2\,.$$

Therefore, we can write

$$\mathscr{L}_{\hbar}^{S} = -4\hbar^2 \partial_z \partial_{\overline{z}} + \hbar B + 4\hbar(2\partial_z \varphi + f'(z))\partial_{\overline{z}}\,,$$

and consider its complexified extension

$$\boxed{\mathscr{L}_{\hbar}^{S} = \hbar \tilde{v}(z,w)\partial_w + \hbar \tilde{B} - 4\hbar^2 \partial_z \partial_w\,, \quad \tilde{v}(z,w) = 8\partial_z \tilde{\varphi}(z,w) + 4f'(z)}\,,$$

acting on analytic functions of $(z, w) \in \mathbb{C}^2$.

First transport equation

The first transport equation, obtained by gathering the terms of order $\hbar$, is

$$\boxed{(\tilde{v}(z,w)\partial_w + \tilde{B}(z,w) - \mu_0)\tilde{a}_0 = 0}\ .$$

Finding μ_0 and f

Let us for now assume that f is given and let $\underline{w}$ be the unique (holomorphic and local) solution of

$$8\partial_z\tilde{\varphi}(z,\underline{w}(z)) + 4f'(z) = 0\,.$$

We deduce that the transport equation has solutions if and only if the exists $\ell \in \mathbb{N}$ such that

$$\tilde{B}(z,\underline{w}(z)) - \mu_0 = -\ell\partial_w\tilde{v}(z,\underline{w}(z))\,.$$

But, from the definition of $\tilde{v}$, this means

$$\boxed{\mu_0 = (2\ell+1)\tilde{B}(z,\underline{w}(z))}\,.$$

Since μ_0 is a constant, we deduce that $\mu_0 = (2\ell+1)b_0$ and

$$\boxed{\tilde{B}(z,\underline{w}(z)) = b_0}\,.$$

Finding μ_0 and f

We choose $\underline{w}(z) = w(z)$, where $w(z)$ is given by a previous lemma. With this choice for $\underline{w}$, we define f as the unique function such that $f(0) = 0$ and

$$\boxed{f'(z) = -2\partial_z \varphi(z, w(z))}\,.$$

Solving the transport equation

We notice that

$$\frac{\tilde{B}(z,w)-b_0}{8\partial_z\varphi(z,w)+4f'(z)}$$

defines a holomorphic function near $(0,0)$. The solutions of the transport equation (with $\mu_0=(2\ell+1)b_0$) have to take the form

$$\tilde{a}_0(z,w)=\mathscr{A}_0(z)(w-w(z))^{\ell}J_\ell(z,w)\,,$$

where

$$J_\ell(z,w)=\exp\left[-\int_{w(z)}^{w}\frac{\tilde{B}-\mu_0}{\tilde{v}}(z,w')+\frac{\ell}{w'-w(z)}\mathrm{d}w'\right].$$

The function $\mathscr{A}_0$ is a holomorphic function to be determined. We take $\ell=0$.

Second transport equation

The equation obtained by gathering the terms in $\hbar^2$ can be written as

$$\boxed{(\tilde{v}(z,w)\partial_w + \tilde{B}(z,w) - \mu_0)\tilde{a}_1 = (\mu_1 + 4\partial_z\partial_w)\,\tilde{a}_0}\,.$$

Effective equation

This equation has solutions if and only if

$$(\mu_1 + 4\partial_z\partial_w)\, \tilde{a}_0(z, w(z)) = 0\,.$$

This means that

$$\boxed{4\mathscr{A}_0'(z)\partial_w J(z, w(z)) + [\mu_1 + 4\partial_w\partial_z J(z, w(z))]\, \mathscr{A}_0(z) = 0}\,.$$

From the definition of J and Taylor expansions, we get

$$4\partial_w J(z, w(z)) \underset{z\to 0}{\sim} -2\frac{\sqrt{\alpha\gamma}}{b_0} z\,, \quad 4\partial_w \partial_z J(0,0) = -\frac{(\sqrt{\alpha}+\sqrt{\gamma})^2}{2b_0}\,.$$

We get that there exists $\ell \in \mathbb{N}$ such that

$$\mu_1 = 2\ell \frac{\sqrt{\alpha\gamma}}{b_0} + \frac{(\sqrt{\alpha}+\sqrt{\gamma})^2}{2b_0} .$$

Then, we can write $\mathscr{A}_0(z) = cz^\ell \widehat{\mathscr{A}_0}(z)$, where $\widehat{\mathscr{A}_0}(z)$ is determined with $\widehat{\mathscr{A}_0}(0) = 1$. The constant c is a normalization constant, we choose $c = 1$.
The solutions take the form

$$\tilde{a}_1(z,w) = \hat{a}_1(z,w) + \mathscr{A}_1(z)J(z,w) ,$$

where $\mathscr{A}_1$ remains to be determined.

We get that there exists $\ell \in \mathbb{N}$ such that

$$\mu_1 = 2\ell \frac{\sqrt{\alpha\gamma}}{b_0} + \frac{(\sqrt{\alpha}+\sqrt{\gamma})^2}{2b_0}\,.$$

Then, we can write $\mathscr{A}_0(z) = cz^\ell \widehat{\mathscr{A}_0}(z)$, where $\widehat{\mathscr{A}_0}(z)$ is determined with $\widehat{\mathscr{A}_0}(0) = 1$. The constant c is a normalization constant, we choose $c = 1$.
The solutions take the form

$$\tilde{a}_1(z, w) = \hat{a}_1(z, w) + \mathscr{A}_1(z)J(z, w)\,,$$

where $\mathscr{A}_1$ remains to be determined.

This procedure can be continued at any order.

Merci de votre attention !

ANNALES
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