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τὸ αὐτὸ νοεῖν ἐστιν τε καὶ εἶναι

Παρμενίδης

Prologue

Pourquoi, un jour, décidons-nous de changer le cours de notre vie, alors que, la veille encore, nous étions installés dans de confortables habitudes ? Je crois que c'est en partie à cette question que mon existence est une réponse. Ce manuscrit contient de nombreux travaux mathématiques auxquels j'ai contribué, et pourtant, je ne suis pas mathématicien. Il n'est en fait que l'ombre d'une autre démarche. La motivation qui m'anime, sans doute, est poétique : créer des mondes par des mots. C'est peut-être pour cela qu'on ne saurait facilement la caractériser. Ainsi, pour faire une analogie, lorsque j'ai commencé à apprendre le piano et le solfège et que j'ai évoqué cette nouvelle activité auprès de François Castella, ce dernier m'a posé une question qui se ramenait finalement à celle-ci : pourquoi ? Et j'ai esquivé. Je fais de la recherche mathématique pour la même raison que je fais du piano ou que je m'intéresse à la philosophie. De même, j'ai découvert cette année certains grands classiques littéraires, comme Don Quichotte, Madame Bovary, La Confusion des Sentiments, Le Lys dans la Vallée ou La Recherche du Temps Perdu. Toutes ces œuvres nous parlent de la même chose. Ce sont des variations géniales sur un même thème : comment les images qu'on projette sur le monde nous en font-elles perdre la saveur ? En particulier, le génie proustien nous montre l'essence de l'art dans les sensations à travers lesquelles s'engouffre notre mémoire involontaire. J'ai décidé de faire du piano en écoutant, dissimulé dans l'encadrement d'une porte, Ari Laptev qui jouait sur celui qui est installé à Mittag-Leffler ; l'espace d'un instant, je me suis souvenu d'un autre piano, d'un autre lieu dont je sentais encore l'humidité, et d'autres mains qui m'avaient appris à aimer la musique ; immédiatement, je flottais là-bas, volant à travers une cour intérieure obscure, passant devant un atelier, et m'envolant au premier étage, au-dessus des vieilles marches en bois, pour rejoindre un ami, assis au piano, tandis que l'eau chauffait dans la cuisine. Alors, pourquoi pas ? me suis-je dit. Peut-être trouverai-je ce qu'il y a de si spécial dans ce souvenir si j'apprends à jouer. J'ai consacré sept ans au Laplacien magnétique et à ses cousins, en suivant la même démarche : laisser se déployer les sentiments, les souvenirs, les pensées, peu importe où cela mène, tant qu'on en sent la nécessité, suivant ainsi les conseils de Rilke au jeune poète. Il y a une grande similitude entre la pensée de Platon et celle de Proust, notamment dans leur rapport à la mémoire. Dans le mythe d'Er, la plupart des hommes se proposent comme avenir leur vie passée, bien souvent en interposant devant leur regard des images sans âme (l'ambition, le pouvoir, la richesse, l'amour, etc.). De même, le narrateur de la Recherche perd son temps à aimer des femmes comme il aimait sa mère : avec anxiété et insatisfaction. Dans les deux cas, une mémoire solidifiée et prête pour l'action surnage et empêche les hommes de voir leur vérité. Dans les deux cas, c'est la métamorphose du sujet à travers la mémoire involontaire (une compréhension intuitive totale) qui l'amène à choisir une vie bonne ou une vie de romancier. La véritable nécessité de cette mémoire, ce n'est pas celle dont une technique parfaite rend compte en coïncidant avec une vérité logique ; c'est bien plutôt celle qui naît de l'aléa des sentiments et qui s'impose à nous. Ce n'est qu'ainsi que je comprends l'activité de chercheur ou que je comprends qu'on puisse aimer plusieurs thèmes de recherche (ou plusieurs personnes : c'est la même chose, non ?), simultanément.

Ainsi s'achève la véritable description de mes travaux et ici commencent mes remerciements. D'abord, je souhaiterais particulièrement remercier Maria Esteban (en lui souhaitant un prompt rétablissement), Gilles Lebeau et Jan Philip Solovej pour avoir accepté de faire un rapport sur mes travaux. Ensuite, je remercie vivement Erwan Faou, Clotilde Fermanian-Kammerer, Bernard Helffer, Frédéric Hérau et San Vũ Ngọc pour leur présence dans mon jury. J'aurais, bien sûr, quelques mentions spéciales à ajouter. Ainsi, je dois à Bernard de m'avoir présenté le Laplacien magnétique et quelques uns de ses secrets ; mais cela a surtout été une aventure humaine de l'avoir rencontré. J'ai un immense plaisir à discuter avec lui et j'ai cru comprendre que nous parlions souvent la même langue : l'intuition. En ce qui concerne Frédéric, je me souviens qu'il m'a dit un jour avec un léger sourire que j'étais plutôt un bon vivant : je le soupçonne d'en être un aussi et je le remercie pour sa bienveillance et les petits concerts de piano improvisés pendant nos pauses WKB. Enfin, je voudrais exprimer ma gratitude à San : sa constance et sa tranquillité m'ont souvent inspiré, pas seulement en ce qui concerne la géométrie symplectique et les formes normales.

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Valete !

À mes amours nécessaires

Contents

CHAPTER 1

A magnetic story

Γνῶθι σεαυτόν.

All this magnetic story is based on the book in preparation [R14b]. This chapter is an informal introduction which points out some connections between the different problems analyzed in the present work. We also provide a detailed description of the contents of this dissertation in Section 3.

1. The realm of $\lambda_1(h)$

1.1. Once upon a time... Let us present two reasons which lead to the analysis of the magnetic Laplacian.

The first motivation arises in the mathematical theory of superconductivity. A model for this theory (see [126]) is given by the Ginzburg-Landau functional:

$$\mathcal{G}(\psi, \mathbf{A}) = \int_\Omega |(-i\nabla + \kappa\sigma\mathbf{A})\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 \, dx + \kappa^2 \int_\Omega |\sigma\nabla \times \mathbf{A} - \sigma\mathbf{B}|^2 \, dx,$$

where $\Omega \subset \mathbb{R}^d$ is the place occupied by the superconductor, ψ is the so-called order parameter ($|\psi|^2$ is the density of Cooper pairs), $\mathbf{A}$ is a magnetic potential and $\mathbf{B}$ the applied magnetic field. The parameter κ is characteristic of the sample (the superconductors of type II are such that $\kappa >> 1$) and σ corresponds to the intensity of the applied magnetic field. Roughly speaking, the question is to determine the nature of the minimizers. Are they normal, that is $(\psi, \mathbf{A}) = (0, \mathbf{F})$ with $\nabla \times \mathbf{F} = \mathbf{B}$ (and $\nabla \cdot \mathbf{F} = 0$), or not? We can mention the important result of Giorgi-Phillips [60] which states that, if the applied magnetic field does not vanish, then, for σ large enough, the normal state is the unique minimizer of $\mathcal{G}$ (with the divergence free condition). When analyzing the local minimality of $(0, \mathbf{F})$, we are led to compute the Hessian of $\mathcal{G}$ at $(0, \mathbf{F})$ and to analyze the positivity of:

$$(-i\nabla + \kappa\sigma\mathbf{A})^2 - \kappa^2.$$

For further details, we refer to the book by Fournais and Helffer [52] and to the papers by Lu and Pan [100, 101]. Therefore the theory of superconductivity leads to investigate the lowest eigenvalue $\lambda_1(h)$ of the Neumann realization of the *magnetic Laplacian*, that is $(-ih\nabla + \mathbf{A})^2$, where $h > 0$ is small (κ is assumed to be large).

The second motivation is to understand at which point there is an analogy between the electric Laplacian $-h^2\Delta + V(x)$ and the magnetic Laplacian $(-ih\nabla + \mathbf{A})^2$. For instance, in the one dimensional electric case, when V admits a unique and non-degenerate minimum at 0 and satisfies $\liminf_{|x|\to+\infty} V(x) > V(0)$, we know that the n-th eigenvalue $\lambda_n(h)$ exists and satisfies:

$$\lambda_n(h) = V(0) + (2n-1)\sqrt{\frac{V''(0)}{2}}h + O(h^2). \tag{1.1.1}$$

Therefore a natural question arises:

"Are there similar results to (1.1.1) in pure magnetic cases?"

In order to answer this question, this dissertation presents a theory of the *Magnetic Harmonic Approximation*. Concerning the Schrödinger equation in presence of magnetic field the reader may consult [5] (see also [29]) and the surveys [110], [44] and [73].

Jointly with (1.1.1) it is well-known that we can perform WKB constructions for the electric Laplacian (see the book of Dimassi and Sjöstrand [35, Chapter 3]). Unfortunately such constructions do not seem to be possible *in full generality* for the pure magnetic case (see the course of Helffer [66, Section 6] and the paper by Martinez and Sordoni [107]) and the naive localization estimates of Agmon are no more optimal (see [83], the paper by Erdős [42] or the papers by Nakamura [113, 114]). In the magnetic situation, accurate semiclassical expansions of the eigenvalues and eigenfunctions (shortly called eigenpairs) are difficult to obtain. In fact, the more we know about the expansion of the eigenpairs, the more we can estimate the tunnel effect in the spirit of the electric tunnel effect of Helffer and Sjöstrand (see for instance [81, 82] and also the papers by Simon [127, 128]) when there are symmetries. Estimating the magnetic tunnel effect is still a widely open question directly related to the approximation of the eigenfunctions (see [83] and [22] for electric tunneling in presence of magnetic field and [12] in the case with corners). Hopefully the main philosophy living throughout this dissertation will prepare the future investigations on this fascinating subject. In particular we provide the first examples of magnetic WKB constructions inspired by the theory developed in [BHR14]. Anyway this dissertation proposes a change of perspective in the study of the magnetic Laplacian. In fact, during the past decades, the philosophy behind the spectral analysis was essentially variational. Many papers dealt with the construction of *quasimodes* used as test functions for the quadratic form associated with the magnetic Laplacian. In any case the attention was focused on the functions of the domain more than on the operator itself. In this dissertation we systematically try to inverse the point of view: the main problem is no more to find *appropriate quasimodes* but an *appropriate (and sometimes microlocal) representation of the operator*. By doing this we will partially leave the min-max principle and the variational theory for the spectral theorem and the microlocal and hypoelliptic spirit.

1.2. Definitions. Let Ω be a Lipschitzian domain in $\mathbb{R}^d$. Let us consider a smooth vector potential $\mathbf{A} = (A_1, \ldots, A_d)$ on $\overline{\Omega}$. We consider the 1-form:

$$\omega_{\mathbf{A}} = \sum_{k=1}^{d} A_k \, \mathrm{d}x_k.$$

We introduce the exterior derivative of $\omega_{\mathbf{A}}$:

$$\sigma_{\mathbf{B}} = \mathrm{d}\omega_{\mathbf{A}} = \sum_{j<k} B_{j,k} \, \mathrm{d}x_j \wedge \mathrm{d}x_k.$$

In dimension two, the only coefficient is $B_{12} = \partial_{x_1} A_2 - \partial_{x_2} A_1$. In dimension three, the magnetic field is defined as:

$$\mathbf{B} = (B_1, B_2, B_3) = (B_{23}, -B_{13}, B_{12}) = \nabla \times \mathbf{A}.$$

We will discuss in this dissertation the spectral properties of some self-adjoint realizations of the magnetic operator:

$$\mathfrak{L}_{h,\mathbf{A},\Omega} = \sum_{k=1}^{d} (-ih\partial_k + A_k)^2,$$

where $h > 0$ is a parameter (related to the Planck constant). We notice the fundamental property, called gauge invariance:

$$e^{-i\phi}(-i\nabla + \mathbf{A})e^{i\phi} = -i\nabla + \mathbf{A} + \nabla\phi$$

so that:

$$e^{-i\phi}(-i\nabla + \mathbf{A})^2 e^{i\phi} = (-i\nabla + \mathbf{A} + \nabla\phi)^2,$$

where $\phi \in \mathsf{H}^1(\Omega, \mathbb{R})$.

1.3. A fascination for $\lambda_1(h)$. In the last fifteen years many papers dealt with the asymptotic expansions of the *first eigenvalue* of the magnetic Laplacian. Let us describe some of these results.

1.3.1. *Constant magnetic field.* In dimension two the constant magnetic field case is treated when Ω is a disk (with Neumann condition) by Bauman, Phillips and Tang in [7] (see [9] and [43] for the Dirichlet case). In particular, they prove a two terms expansion of the first eigenvalue:

$$\lambda_1(h) = \Theta_0 h - \frac{C_1}{R} h^{3/2} + o(h^{3/2}),$$

where $\Theta_0 \in (0, 1)$ and $C_1 > 0$ are universal constants. This result, which was conjectured in [8, 34], is generalized to smooth and bounded domains by Helffer and Morame in [75] where it is proved that:

$$\lambda_1(h) = \Theta_0 h - C_1 \kappa_{max} h^{3/2} + o(h^{3/2}), \tag{1.1.2}$$

where κ_{max} is the maximal curvature of the boundary. Let us emphasize that, in these papers, the authors are only concerned by the first terms of the asymptotic expansion of

$\lambda_1(h)$. In the case of smooth domains the complete asymptotic expansion of all the eigenvalues is done by Fournais and Helffer in [51]. When the boundary is not smooth, we may mention the papers of Jadallah and Pan [87, 118]. In the semiclassical regime, we refer to the papers of Bonnaillie-Noël, Dauge and Fournais [10, 11, 14] where perturbation theory is used in relation with the estimates of Agmon. For numerical investigations the reader may consider the paper [12].

In dimension three the constant magnetic field case (with intensity 1) is treated by Helffer and Morame in [77] under generic assumptions on the (smooth) boundary of Ω:

$$\lambda_1(h) = \Theta_0 h + \hat{\gamma}_0 h^{4/3} + o(h^{4/3}),$$

where the constant $\hat{\gamma}_0$ is related to the magnetic curvature of a curve in the boundary along which the magnetic field is tangent to the boundary. The case of the ball is analyzed in details by Fournais and Persson in [53].

1.3.2. *Variable magnetic field.* The case of variable magnetic fields is the core of this dissertation. This case can be motivated by anisotropic superconductors (see for instance [25, 2]) or the liquid crystal theory (see [78, 79, R10a, R10b]). Nevertheless we will see that the variable situation has an interest in itself and will lead to considerations that may apply to the constant magnetic field case as well. One of the main (and not so naive) ideas in this dissertation is that a variable geometry with a constant magnetic field can be transformed into a constant geometry with an effective variable magnetic field (and even an electric field in the semiclassical limit). Let us now recall some personal results which gave birth to the intuitions pervading this dissertation.

For the case with a non vanishing variable magnetic field, we refer to [100, R09] for the first terms of the lowest eigenvalue. In particular the paper [R09] provides (under a generic condition) an asymptotic expansion with two terms in the form:

$$\lambda_1(h) = \Theta_0 b' h + C_1^{\mathsf{2D}}(\mathbf{x}_0, \mathbf{B}, \partial\Omega) h^{3/2} + o(h^{3/2}),$$

where $C_1^{\mathsf{2D}}(\mathbf{x}_0, \mathbf{B}, \partial\Omega)$ depends on the geometry of the boundary and on the magnetic field at $\mathbf{x}_0$ and where $b' = \min_{\partial\Omega} B = B(\mathbf{x}_0)$. When the magnetic field vanishes, the first analysis of the lowest eigenvalue is due to Montgomery in [111] followed by Helffer and Morame in [74] (see also [119, 68, 70]).

In dimension three (with Neumann condition on a smooth boundary), the first term of $\lambda_1(h)$ is given by Lu and Pan in [101]. The next terms in the expansion are investigated in [R10c] where we can find in particular an upper bound in the form

$$\lambda_1(h) \le \|\mathbf{B}(\mathbf{x}_0)\| \mathfrak{s}(\theta(\mathbf{x}_0)) h + C_1^{\mathsf{3D}}(\mathbf{x}_0, \mathbf{B}, \partial\Omega) h^{3/2} + C_2^{\mathsf{3D}}(\mathbf{x}_0, \mathbf{B}, \partial\Omega) h^2 + C h^{5/2},$$

where $\mathfrak{s}$ is a spectral invariant defined in the next section, $\theta(\mathbf{x}_0)$ is the angle of $\mathbf{B}(\mathbf{x}_0)$ with the boundary at $\mathbf{x}_0$ and the constants $C_j^{\mathsf{3D}}(\mathbf{x}_0, \mathbf{B}, \partial\Omega)$ are related to the geometry and the magnetic field at $\mathbf{x}_0 \in \partial\Omega$. Let us finally mention the recent paper by Bonnaillie-Noël-Dauge-Popoff [13] which establishes a one term asymptotics in the case of Neumann boundaries with corners.

1.4. Some model operators. It turns out that the results recalled in Section 1.3 are related to many model operators. Let us introduce some of them.

1.4.1. *De Gennes operator.* The analysis of the magnetic Laplacian with Neumann condition on $\mathbb{R}^2_+$ makes the so-called de Gennes operator to appear. We refer to [32] where this model is studied in details (see also [52]). For $\zeta \in \mathbb{R}$, we consider the Neumann realization on $\mathsf{L}^2(\mathbb{R}_+)$ of

$$\mathfrak{L}^{[0]}_{\zeta} = D_t^2 + (\zeta - t)^2. \tag{1.1.3}$$

We denote by $\nu^{[0]}_1(\zeta)$ the lowest eigenvalue of $\mathfrak{L}^{[0]}_{\zeta}$. It is possible to prove that the function $\zeta \mapsto \nu^{[0]}_1(\zeta)$ admits a unique and non-degenerate minimum at a point $\zeta^{[0]}_0 > 0$, shortly denoted by ζ_0, and that we have

$$\Theta_0 := \min_{\xi\in\mathbb{R}} \nu^{[0]}_1(\zeta) \in (0,1). \tag{1.1.4}$$

1.4.2. *Montgomery operator.* Let us now introduce another important model. This one was introduced by Montgomery in [111] to study the case of vanishing magnetic fields in dimension two (see also [119] and [77, Section 2.4]). This model was revisited by Helffer in [67], generalized by Helffer and Persson in [80] and Fournais and Persson in [54]. The Montgomery operator with parameter $\zeta \in \mathbb{R}$ is the self-adjoint realization on $\mathbb{R}$ of:

$$\mathfrak{L}^{[1]}_{\zeta} = D_t^2 + \left(\zeta - \frac{t^2}{2}\right)^2. \tag{1.1.5}$$

1.4.3. *Popoff operator.* The investigation of the magnetic Laplacian on dihedral domains (see [121]) leads to the analysis of the Neumann realization on $\mathsf{L}^2(\mathcal{S}_\alpha, \,\mathrm{d}t\,\mathrm{d}z)$ of:

$$\mathfrak{L}^{\mathsf{e}}_{\alpha,\zeta} = D_t^2 + D_z^2 + (t-\zeta)^2, \tag{1.1.6}$$

where $\mathcal{S}_\alpha$ is the sector with angle α,

$$\mathcal{S}_\alpha = \left\{(t,z) \in \mathbb{R}^2 : |z| \leq t \tan\left(\frac{\alpha}{2}\right)\right\}.$$

1.4.4. *Lu-Pan operator.* Let us present a last model operator appearing in dimension three in the case of a smooth Neumann boundary (see [101, 76, BDPR12]). We consider the half-plane,

$$\mathbb{R}^2_+ = \{(s,t) \in \mathbb{R}^2,\ t > 0\}$$

and we introduce the self-adjoint Neumann realization on $\mathbb{R}^2_+$ of the Schrödinger operator $\mathfrak{L}^{\mathsf{LP}}_{\theta}$ with potential V_θ:

$$\mathfrak{L}^{\mathsf{LP}}_{\theta} = -\Delta + V_\theta = D_s^2 + D_t^2 + V_\theta, \tag{1.1.7}$$

where V_θ is defined for any $\theta \in (0, \frac{\pi}{2})$ by

$$V_\theta : (s,t) \in \mathbb{R}^2_+ \longmapsto (t\cos\theta - s\sin\theta)^2.$$

We can notice that V_θ reaches its minimum 0 all along the line $t\cos\theta = s\sin\theta$, which makes the angle θ with $\partial\mathbb{R}^2_+$. We denote by $\mathfrak{s}_1(\theta)$ or simply $\mathfrak{s}(\theta)$ the infimum of the spectrum of $\mathfrak{L}^{\mathsf{LP}}_\theta$. In [**52**] (and [**76, 101**]), it is proved that $\mathfrak{s}$ is analytic and strictly increasing on $\left(0, \frac{\pi}{2}\right)$.

2. A connection with waveguides

2.1. Existence of a bound state of $\mathfrak{L}^{\mathsf{LP}}_\theta$. Among other things one can prove the following lemma (see [**76, 101**]).

Lemma 1.1. *For all $\theta \in \left(0, \frac{\pi}{2}\right)$ there exists an eigenvalue of $\mathfrak{L}^{\mathsf{LP}}_\theta$ below the essential spectrum which equals* $[1, +\infty)$.

A classical result combining an estimate of Agmon (cf. [**1**]) and a theorem due to Persson (cf. [**120**]) implies that the corresponding eigenfunctions are localized near $(0,0)$. This result is slightly surprising since the existence of the discrete spectrum is related to the association between the Neumann condition and the partial confinement of V_θ. After translation and rescaling, we are led to a new operator:

$$hD_s^2 + D_t^2 + (t - \zeta_0 - sh^{1/2})^2 - \Theta_0,$$

where $h = \tan\theta$. Then one can reduce the (semiclassical) analysis to the so-called *Born-Oppenheimer* approximation (see for instance [**103**]):

$$hD_s^2 + \nu_1^{[0]}(\zeta_0 + sh^{1/2}) - \Theta_0.$$

This last operator is very easy to analyze with the classical theory of the harmonic approximation and we get (see [**BDPR12**]):

Theorem 1.2. *The lowest eigenvalues of $\mathfrak{L}^{\mathsf{LP}}_\theta$ admit the following expansions:*

$$\mathfrak{s}_n(\theta) \underset{\theta\to 0}{\sim} \sum_{j\geq 0} \gamma_{j,n}\theta^j \tag{1.2.1}$$

with $\gamma_{0,n} = \Theta_0$ et $\gamma_{1,n} = (2n-1)\sqrt{\frac{(\nu_1^{[0]})''(\zeta_0)}{2}}$.

2.2. A result of Duclos and Exner. Figure 1 can make us think to a *broken waveguide*. Indeed, if one uses the Neumann condition to symmetrize $\mathfrak{L}^{\mathsf{LP}}_\theta$ and if one replaces the confinement property of V_θ by a Dirichlet condition, we are led to the situation described in Figure 2. This heuristic comparison reminds the famous paper [**37**] where Duclos and Exner introduce a definition of standard (and smooth) waveguides and

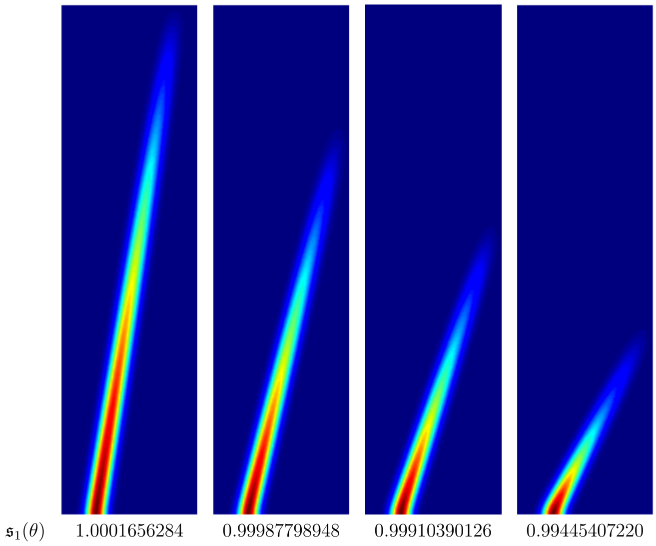


FIGURE 1. First eigenfunction of $\mathfrak{L}_\theta^{\mathsf{LP}}$ for $\theta = \vartheta\pi/2$ with $\vartheta = 0.9$, 0.85, 0.8 et 0.7.

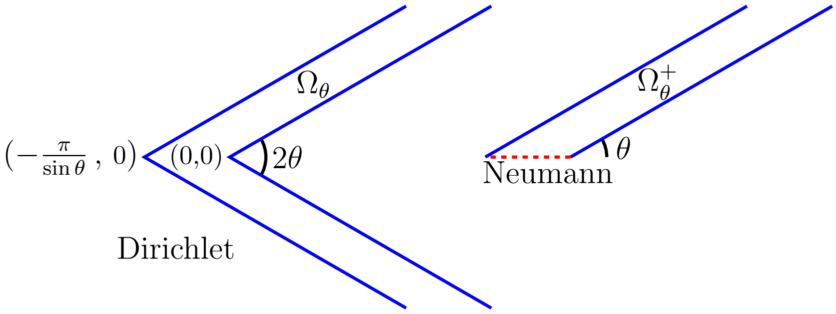


FIGURE 2. Waveguide with corner Ω_θ and half-waceguide Ω_θ^+.

perform a spectral analysis. For example, in dimension two (see Figure 3), a waveguide of width ε is determined by a smooth curve $s \mapsto c(s) \in \mathbb{R}^2$ as the subset of $\mathbb{R}^2$ given by:

$$\{c(s) + t\boldsymbol{n}(s), \quad (s,t) \in \mathbb{R} \times (-\varepsilon, \varepsilon)\},$$

where $\boldsymbol{n}(s)$ is the normal to the curve $c(\mathbb{R})$ at the point $c(s)$.

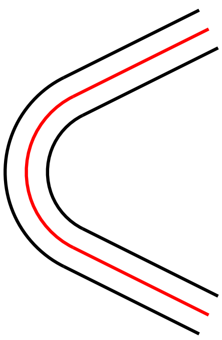


FIGURE 3. Waveguide

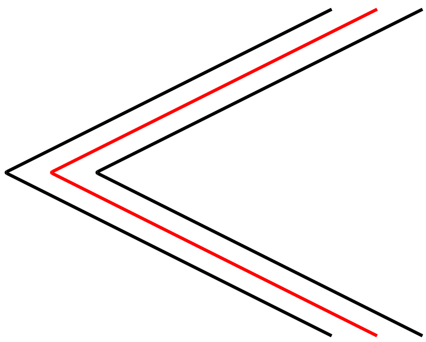


FIGURE 4. Broken guide

Assuming that the waveguide is straight at infinity but not everywhere, Duclos and Exner prove that there is always an eigenvalue below the essential spectrum (in the case of a circular cross section in dimensions two and three). Let us notice that the essential spectrum is $[\lambda, +\infty)$ where λ is the lowest eigenvalue of the Dirichlet Laplacian on the cross section. The proof of the existence of discrete spectrum is elementary and relies on the min-max principle. Letting $\psi \in H^1_0(\Omega)$:

$$q(\psi) = \int_\Omega |\nabla\psi|^2 \, \mathrm{d}\mathbf{x},$$

it is enough to find ψ_0 such that $q(\psi_0) < \lambda \|\psi_0\|_{\mathsf{L}^2(\Omega)}$. Such a function can be constructed by considering a perturbed Weyl sequence associated with λ.

2.3. Waveguides and magnetic fields. Bending a waveguide induces discrete spectrum below the essential spectrum, but what about twisting a waveguide? This question arises for instance in the papers [**92**, **95**, **41**] where it is proved that twisting a waveguide plays against the existence of the discrete spectrum. In the case without curvature, the quadratic form is defined for $\psi \in \mathsf{H}^1_0(\mathbb{R}\times\omega)$ by:

$$q(\psi) = \|\partial_1\psi - \rho(s)(t_3\partial_2 - t_2\partial_3)\psi\|^2 + \|\partial_2\psi\|^2 + \|\partial_3\psi\|^2,$$

where $s \mapsto \rho(s)$ represents the effect of twisting the cross section ω and (t_2, t_3) are coordinates in ω. From a heuristic point of view, the twisting perturbation seems to act "as" a magnetic field. This leads to the natural question:

"Is the spectral effect of a torsion the same as the effect of a magnetic field?"

If the geometry of a waveguide can formally generate a magnetic field, we can conversely wonder if a magnetic field can generate a waveguide. This remark partially appears in [**36**] where the discontinuity of a magnetic field along a line plays the role of a waveguide. More generally it turns out that, when the magnetic field cancels along a curve, this curve becomes an effective waveguide.

3. Organization of the dissertation

3.1. Spectral analysis of model operators and spectral reductions. Chapter 2 deals with model operators. This notion of model operators is fundamental in the theory of the magnetic Laplacian. We have already introduced some important and historical examples. There are essentially two natural ways to meet reductions to model operators. The first approach can be done thanks to a (space) partition of unity which reduces the spectral analysis to the one of localized and simplified models (we straighten the geometry and freeze the magnetic field). The second approach, which involves an analysis in the phase space, is to identify the possible different scales of the problem, that is the fast and slow variables. This often involves an investigation in the microlocal spirit: we shall analyze the properties of symbols and deduce a microlocal reduction to a spectral problem in lower dimension. In Chapter 2 we provide explicit examples of models and provide their spectral analysis. In Chapter 2, Section 1 we introduce a model which is fundamental to describe the effect of conical singularities of the boundary on the magnetic eigenvalues (see [**BR13a, BR14**]). This is an example which is provided by the first kind of approach (freeze the geometry and the magnetic field). It will turn out that part of the spectral analysis of this model can be done in the spirit of the second approach when the angle of the cone goes to zero (different scales and dimensional reduction). In Chapter 2, Section 2 we present a model related to vanishing magnetic fields in dimension two. Due to an inhomogeneity of the magnetic operator, this other model leads to a microlocal reduction and therefore to the investigation of an effective symbol (see [**BR13b, R14a**]). In fact, the example of Section 2 can lead to a more general framework. In Chapter 2, Section 3 we provide a general and elementary theory of the "magnetic Born-Oppenheimer approximation" which is a systematic semiclassical reduction to model operators (under generic assumptions on some effective symbols). We also provide the first known pure magnetic WKB constructions (see [**BHR14**]).

3.2. Normal forms philosophy and the magnetic semi-excited states. As we have seen there is a non trivial connection between the discrete spectrum, the possible magnetic field and the possible boundary. In fact *normal form* procedures are often deeply rooted in the different proofs, not only in the semiclassical framework. We present in Chapter 3 the results of four studies [**DoR13**], [**R12**], [**PR13**], [**RVN14**]. These studies are concerned by the semiclassical asymptotics of the magnetic eigenvalues and eigenfunctions.

3.2.1. *Towards the magnetic semi-excited states.* We now describe the philosophy of the proofs of asymptotic expansions for the magnetic Laplacian with respect to a parameter α (which tends to zero and which might be for example the semiclassical parameter). Let us distinguish between the different conceptual levels of the analysis. Our analysis uses the standard construction of quasimodes, localization techniques ("IMS" formula) and *a priori* estimates of Agmon type satisfied by the eigenfunctions. These "standard" tools, which are used in most of the papers dealing with $\lambda_1(\alpha)$, are not enough to investigate $\lambda_n(\alpha)$ due to the spectral splitting arising sometimes in the subprincipal terms. In fact such a fine behavior is the sign of a microlocal effect. In order to investigate this effect, we use normal form procedures *in the spirit of the Egorov theorem*. It turns out that this normal form strategy also strongly simplifies the construction of quasimodes. Once the behavior of the eigenfunctions in the phase space is established, we use the Feshbach-Grushin approach to reduce our operator to an electric Laplacian. Let us comment more in details the whole strategy.

The first step to analyze such problems is to perform an accurate construction of quasimodes and to apply the spectral theorem. In other words we look for pairs (λ, ψ) such that we have $\|(\mathcal{L}_\alpha - \lambda)\psi\| \leq \varepsilon\|\psi\|$. Such pairs are constructed through an homogenization procedure involving different scales with respect to the different variables. In particular the construction uses a formal power series expansion of the operator and an Ansatz in the same form for (λ, ψ). The main difficulty in order to succeed is to choose the appropriate scalings.

The second step aims at giving *a priori* estimates satisfied by the eigenfunctions.These are localization estimates *à la Agmon* (see [**1**]). To prove them one generally needs to have *a priori* estimates for the eigenvalues which can be obtained with a partition of unity and local comparisons with model operators. Then such *a priori* estimates, which are in general not optimal, involve an improvement in the asymptotic expansion of the eigenvalues. If we are just interested in the first terms of $\lambda_1(\alpha)$, these classical tools are enough.

In fact, the major difference with the electric Laplacian arises precisely in the analysis of the spectral splitting between the lowest eigenvalues. Let us describe what is done in [**51**] (dimension two, constant magnetic field, $\alpha = h$) and in [**R13a**] (non constant magnetic field). In [**51, R13a**] quasimodes are constructed and the usual localization estimates are proved. Then the behavior with respect to a phase variable needs to be determined to allow a dimensional reduction. Let us underline here that this phenomenon of phase localization is characteristic of the magnetic Laplacian and is intimately related to the structure of the low lying spectrum. In [**51**] Fournais and Helffer are led to use the pseudo-differential calculus and the Grushin formalism. In [**R13a**] the approach is structurally not the same. In [**R13a**], in the spirit of the Egorov theorem (see [**39, 124, 105**]), we use successive canonical transforms of the symbol of the operator corresponding to unitary transforms (change of gauge, change of variable, Fourier transform) and we reduce the operator, modulo remainders which are controlled thanks to the *a priori* estimates, to an electric Laplacian being in the Born-Oppenheimer form (see [**27, 103**] and more recently [**BDPR12**]). This reduction highlights the crucial idea that the inhomogeneity of the magnetic operator is responsible for its spectral structure, as we can see in [**DoR13**], [**PR13**].

3.2.2. *Birkhoff normal form.* As we suggested above, our magnetic normal forms are close to the Birkhoff procedure and it is rather surprising that it has never been implemented to describe the effect of magnetic fields on the low lying eigenvalues of the magnetic Laplacian. A reason might be that, compared to the case of a Schrödinger operator with an electric potential, the pure magnetic case presents a specific feature: the symbol "itself" is not enough to generate a localization of the eigenfunctions. This difficulty can be seen in the recent papers by Helffer and Kordyukov [**69**] (dimension two) and [**71**] (dimension three) which treat cases without boundary. In dimension three they provide accurate constructions of quasimodes, but do not establish the semiclassical asymptotic expansions of the eigenvalues which is still an open problem. In dimension two, they prove that if the magnetic field has a unique and non-degenerate minimum, the

j-th eigenvalue admits an expansion in powers of $h^{1/2}$ of the form:

$$\lambda_j(h) \sim h \min_{q\in\mathbb{R}^2} B(q) + h^2(c_1(2j-1)+c_0) + O(h^{5/2}),$$

where c_0 and c_1 are constants depending on the magnetic field. In [**RVN14**], we extend their result by obtaining a complete asymptotic expansion which actually applies to more general magnetic wells and allows to describe larger eigenvalues. In the ongoing work [**HKRVN14**], we extend this strategy to the dimension three.

3.3. The spectrum of waveguides. In Chapter 4 we present some results occurring in the theory of waveguides. In particular we consider the following question (addressed in [**KR13**]):

"What is the spectral influence of a magnetic field on a waveguide ?"

Then, when there is no magnetic field, we would also like to analyze the effect of a corner on the spectrum and present a non smooth version of the result of Duclos and Exner (see [**DaR12**]). For that purpose we also present some results concerning the *semiclassical triangles*.

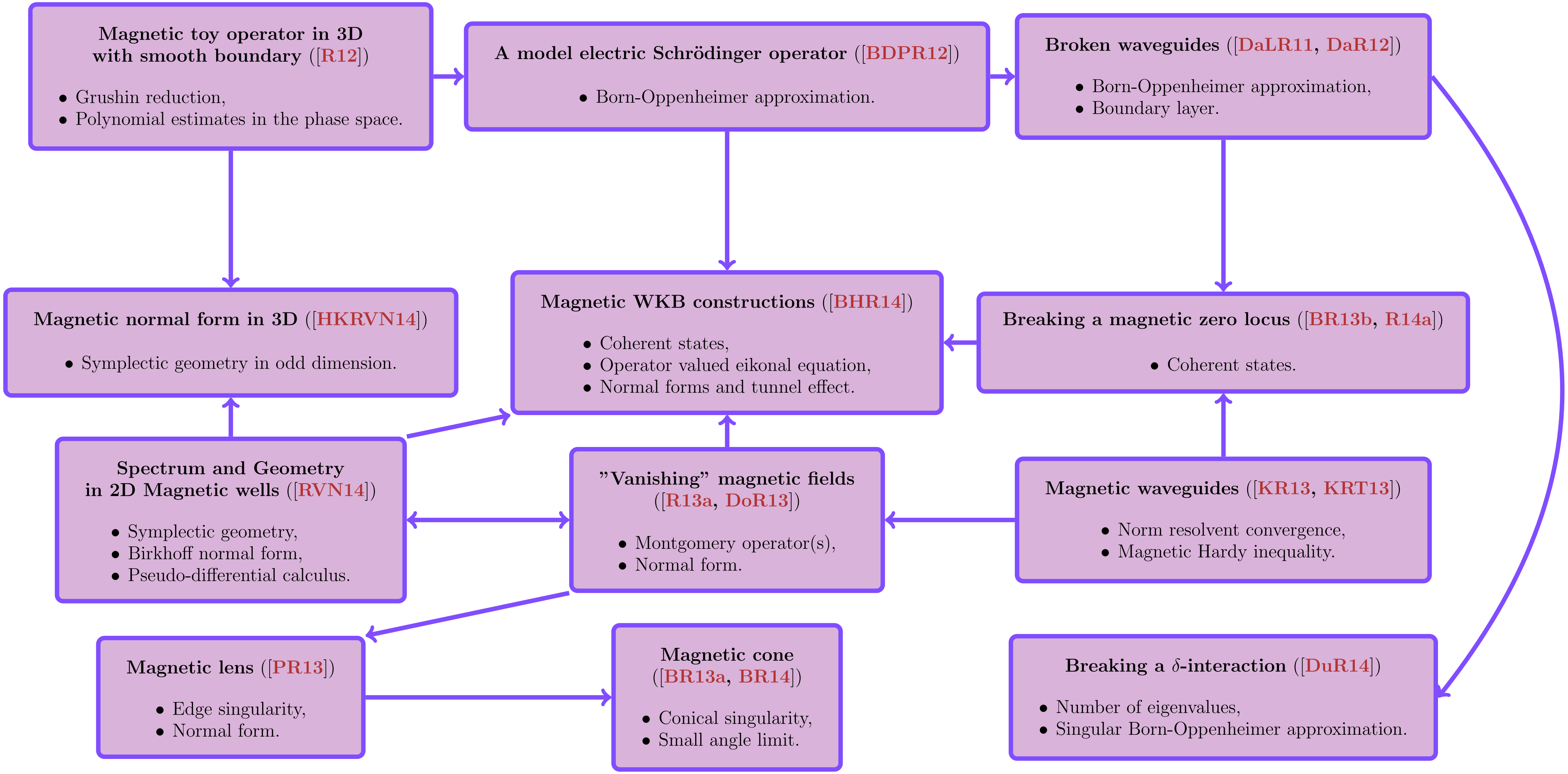


FIGURE 5. Some references and connections

CHAPTER 2

Models and spectral reductions

The soul unfolds itself, like a lotus of countless petals.
The Prophet, Self-Knowledge, Khalil Gibran

In this chapter we introduce two model operators (depending on parameters).

The first one is the Neumann Laplacian on a circular cone of aperture α with a constant magnetic field. This model is quite important in the study of problems with non smooth boundaries in dimension three: this is the simplest case involving a conical singularity. The results presented about this operator are based on the collaborations with V. Bonnaillie-Noël [**BR13a**] and [**BR14**].

The second one appears in dimension two when studying vanishing magnetic fields in the case when the cancellation line of the field intersects the boundary. The results concerning this model are related to [**BR13b**] and [**R14a**].

These models will already give a flavor of the techniques which travel through this dissertation.

Finally, we provide in this chapter a theory of the magnetic Born-Oppenheimer approximation as well as purely magnetic WKB constructions based on the collaboration with V. Bonnailllie-Noël and F. Hérau [**BHR14**].

1. The power of the peaks

We are interested in the low-lying eigenvalues of the magnetic Neumann Laplacian with a constant magnetic field applied to a “ peak ”, i.e. a right circular cone $\mathcal{C}_\alpha$. The right circular cone $\mathcal{C}_\alpha$ of angular opening $\alpha \in (0, \pi)$ (see Figure 1) is defined in the Cartesian coordinates (x, y, z) by

$$\mathcal{C}_\alpha = \{(x, y, z) \in \mathbb{R}^3,\ z > 0,\ x^2 + y^2 < z^2 \tan^2 \tfrac{\alpha}{2}\}.$$

Let $\mathbf{B}$ be the constant magnetic field

$$\mathbf{B}(x, y, z) = (0, \sin\beta, \cos\beta)^\mathsf{T},$$

where $\beta \in \left[0, \frac{\pi}{2}\right]$. We choose the following magnetic potential $\mathbf{A}$:

$$\mathbf{A}(x, y, z) = \frac{1}{2}\mathbf{B} \times \mathbf{x} = \frac{1}{2}(z\sin\beta - y\cos\beta, x\cos\beta, -x\sin\beta)^\mathsf{T}.$$

We consider $\mathfrak{L}_{\alpha,\beta}$ the Friedrichs extension associated with the quadratic form

$$\mathcal{Q}_{\mathbf{A}}(\psi) = \|(-i\nabla + \mathbf{A})\psi\|^2_{\mathsf{L}^2(\mathcal{C}_\alpha)},$$

defined for $\psi \in \mathsf{H}^1_{\mathbf{A}}(\mathcal{C}_\alpha)$ with

$$\mathsf{H}^1_{\mathbf{A}}(\mathcal{C}_\alpha) = \{u \in \mathsf{L}^2(\mathcal{C}_\alpha), (-i\nabla + \mathbf{A})u \in \mathsf{L}^2(\mathcal{C}_\alpha)\}.$$

The operator $\mathfrak{L}_\alpha$ is $(-i\nabla + \mathbf{A})^2$ with domain:

$$\mathsf{H}^2_{\mathbf{A}}(\mathcal{C}_\alpha) = \{u \in \mathsf{H}^1_{\mathbf{A}}(\mathcal{C}_\alpha), (-i\nabla + \mathbf{A})^2 u \in \mathsf{L}^2(\mathcal{C}_\alpha), (-i\nabla + \mathbf{A})u \cdot \mathbf{n} = 0 \text{ on } \partial\mathcal{C}_\alpha\}.$$

We define $\lambda_n(\alpha,\beta)$ as the n-th Rayleigh quotient of $\mathfrak{L}_{\alpha,\beta}$. Let $\psi_n(\alpha,\beta)$ be a normalized associated eigenvector (if it exists).

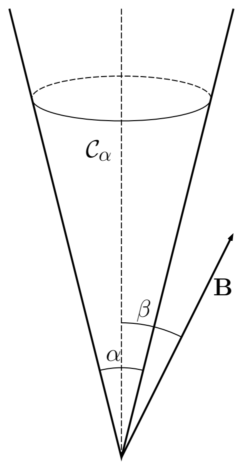


FIGURE 1. Geometric setting.

1.1. Why studying magnetic cones? One of the most interesting results of the last fifteen years is provided by Helffer and Morame in [75] where they prove that the magnetic eigenfunctions, in 2D, concentrates near the points of the boundary where the (algebraic) curvature is maximal, see (1.1.2). This property aroused interest in domains with corners, which somehow correspond to points of the boundary where the curvature becomes infinite (see [87, 118] for the quarter plane and [10, 11] for more general domains). Denoting by $\mathcal{S}_\alpha$ the sector in $\mathbb{R}^2$ with angle α and considering the magnetic Neumann Laplacian with constant magnetic field of intensity 1, it is proved in [10] that, as soon as α is small enough, a bound state exists. Its energy is denoted by $\mu(\alpha)$. An asymptotic expansion at any order is even provided (see [10, Theorem 1.1]):

$$\mu(\alpha) \sim \alpha \sum_{j\geq 0} m_j \alpha^{2j}, \qquad \text{with} \quad m_0 = \frac{1}{\sqrt{3}}. \tag{2.1.1}$$

In particular, this proves that $\mu(\alpha)$ becomes smaller than the lowest eigenvalue of the magnetic Neumann Laplacian in the half-plane with constant magnetic field (with intensity 1), that is:

$$\mu(\alpha) < \Theta_0, \qquad \alpha \in (0, \alpha_0),$$

where Θ_0 is defined in (1.1.4).This motivates the study of dihedral domains (see [**121, 122**]). Another possibility of investigation, in dimension three, is the case of a conical singularity of the boundary. We would especially like to answer the following questions: Can we go below $\mu(\alpha)$ and can we describe the structure of the spectrum when the aperture of the cone goes to zero?

1.2. The magnetic Laplacian in spherical coordinates. Since the spherical coordinates are naturally adapted to the geometry, we consider the change of variable:

$$\Phi(\tau, \theta, \varphi) := (x, y, z) = \alpha^{-1/2}(\tau \cos\theta \sin\alpha\varphi,\ \tau \sin\theta \sin\alpha\varphi,\ \tau \cos\alpha\varphi).$$

This change of coordinates is nothing but a first normal form. We denote by $\mathcal{P}$ the semi-infinite rectangular parallelepiped

$$\mathcal{P} := \{(\tau, \theta, \varphi) \in \mathbb{R}^3,\ \tau > 0,\ \theta \in [0, 2\pi),\ \varphi \in (0, \tfrac{1}{2})\}.$$

Let $\psi \in \mathsf{H}^1_{\mathbf{A}}(\mathcal{C}_\alpha)$. We write $\psi(\Phi(\tau, \theta, \varphi)) = \alpha^{1/4}\tilde{\psi}(\tau, \theta, \varphi)$ for any $(\tau, \theta, \varphi) \in \mathcal{P}$ in these new coordinates and we have

$$\|\psi\|^2_{\mathsf{L}^2(\mathcal{C}_\alpha)} = \int_{\mathcal{P}} |\tilde{\psi}(\tau, \theta, \varphi)|^2\, \tau^2 \sin\alpha\varphi \,\mathrm{d}\tau\,\mathrm{d}\theta\,\mathrm{d}\varphi,$$

and:

$$\mathfrak{Q}_{\mathbf{A}}(\psi) = \alpha \mathcal{Q}_{\alpha,\beta}(\tilde{\psi}),$$

where the quadratic form $\mathcal{Q}_{\alpha,\beta}$ is defined on the transformed form domain $\mathsf{H}^1_{\tilde{\mathbf{A}}}(\mathcal{P})$ by

$$\mathcal{Q}_{\alpha,\beta}(\psi) := \int_{\mathcal{P}} \left(|P_1\psi|^2 + |P_2\psi|^2 + |P_3\psi|^2\right)\,\mathrm{d}\tilde{\mu}, \tag{2.1.2}$$

with the measure

$$\mathrm{d}\tilde{\mu} = \tau^2 \sin\alpha\varphi \,\mathrm{d}t\,\mathrm{d}\theta\,\mathrm{d}\varphi,$$

and:

$$\mathsf{H}^1_{\tilde{\mathbf{A}}}(\mathcal{P}) = \{\psi \in \mathsf{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu}), (-i\nabla + \tilde{\mathbf{A}})\psi \in \mathsf{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})\}.$$

We also have:

$$\begin{aligned}
&P_1 = D_\tau - \tau\varphi\cos\theta\sin\beta,\\
&P_2 = (\tau\sin(\alpha\varphi))^{-1}\left(D_\theta + \frac{\tau^2}{2\alpha}\sin^2(\alpha\varphi)\cos\beta + \frac{\tau^2\varphi}{2}\left(1 - \frac{\sin(2\alpha\varphi)}{2\alpha\varphi}\right)\sin\beta\sin\theta\right),\\
&P_3 = (\tau\sin(\alpha\varphi))^{-1}D_\varphi.
\end{aligned}$$

We consider $\mathcal{L}_{\alpha,\beta}$ the Friedrichs extension associated with the quadratic form $\mathcal{Q}_{\alpha,\beta}$:

$$\begin{aligned}\mathcal{L}_{\alpha,\beta} &= \tau^{-2}(D_\tau - \tau\varphi\cos\theta\sin\beta)\tau^2(D_\tau - \tau\varphi\cos\theta\sin\beta)\\ &\quad+\frac{1}{\tau^2\sin^2(\alpha\varphi)}\left(D_\theta + \frac{\tau^2}{2\alpha}\sin^2(\alpha\varphi)\cos\beta + \frac{\tau^2\varphi}{2}\left(1-\frac{\sin(2\alpha\varphi)}{2\alpha\varphi}\right)\sin\beta\sin\theta\right)^2\\ &\quad+\frac{1}{\alpha^2\tau^2\sin(\alpha\varphi)}D_\varphi\sin(\alpha\varphi)D_\varphi.\end{aligned}$$

We define $\tilde{\lambda}_n(\alpha,\beta)$ the n-th eigenvalue of $\mathcal{L}_{\alpha,\beta}$.

1.3. Spectrum of the magnetic cone in the small angle limit.

1.3.1. *Eigenvalues in the small angle limit.* We aim at estimating the discrete spectrum, if it exists, of $\mathfrak{L}_{\alpha,\beta}$. For that purpose, we shall first determine the bottom of its essential spectrum. From Persson's characterization of the infimum of the essential spectrum, it is enough to consider the behavior at infinity and it is possible to establish the following proposition (see [**BR14**]).

Proposition 2.1. *Let us denote by $\mathsf{sp}_{\mathsf{ess}}(\mathfrak{L}_{\alpha,\beta})$ the essential spectrum of $\mathfrak{L}_{\alpha,\beta}$. We have:*

$$\inf \mathsf{sp}_{\mathsf{ess}}(\mathfrak{L}_{\alpha,\beta}) \in [\Theta_0, 1],$$

where $\Theta_0 > 0$ is defined in (1.1.4).

At this stage we still do not know that discrete spectrum exists. As it is the case in dimension two (see [**10**]) or in the case on the infinite wedge (see [**121**]), there is hope to prove such an existence in the limit $\alpha \to 0$ (see [**BR14**]).

Theorem 2.2. *For all $n \geq 1$, there exist $\alpha_0(n) > 0$ and a sequence $(\gamma_{j,n})_{j\geq 0}$ such that, for all $\alpha \in (0, \alpha_0(n))$, the n-th eigenvalue exists and satisfies:*

$$\lambda_n(\alpha,\beta) \underset{\alpha\to 0}{\sim} \alpha\sum_{j\geq 0}\gamma_{j,n}\alpha^j, \qquad \text{with} \quad \gamma_{0,n} = \frac{\sqrt{1+\sin^2\beta}}{2^{5/2}}(4n-1).$$

Remark 2.3. *In particular the main term is minimum when $\beta = 0$ and in this case Theorem 2.2 states that $\lambda_1(\alpha) \sim \frac{3}{2^{5/2}}\alpha$. We have $\frac{3}{2^{5/2}} < \frac{1}{\sqrt{3}}$ so that the lowest eigenvalue of the magnetic cone goes below the lowest eigenvalue of the two dimensional magnetic sector (see (2.1.1)).*

Remark 2.4. *As a consequence of Theorem 2.2, we deduce that the lowest eigenvalues are simple as soon as α is small enough. Therefore, the spectral theorem implies that the quasimodes constructed in the proof are approximations of the eigenfunctions of $\mathcal{L}_{\alpha,\beta}$. In particular, using the rescaled spherical coordinates, for all $n \geq 1$, there exist $\alpha_n > 0$ and C_n such that, for $\alpha \in (0, \alpha_n)$:*

$$\|\tilde{\psi}_n(\alpha,\beta) - \mathfrak{f}_n\|_{\mathsf{L}^2(\mathcal{P},\mathrm{d}\tilde{\mu})} \leq C_n\alpha^2,$$

where $\mathfrak{f}_n$ (which is β dependent) is related to the n-th Laguerre's function and $\tilde{\psi}_n(\alpha,\beta)$ is the n-th normalized eigenfunction.

In the next two sections we discuss the strategy of the proof of Theorem 2.2.

1.3.2. *Axissymmetric case:* $\beta = 0$. We apply the strategy presented in Chapter 1, Section 3. In this situation, the phase variable that we should understand is the dual variable of θ given by a Fourier series decomposition and denoted by $m \in \mathbb{Z}$. In other words, we make a Fourier decomposition of $\mathcal{L}_{\alpha,0}$ with respect to θ and we introduce the family of 2D-operators $(\mathcal{L}_{\alpha,0,m})_{m\in\mathbb{Z}}$ acting on $\mathsf{L}^2(\mathcal{R}, \mathrm{d}\mu)$:

$$\mathcal{L}_{\alpha,0,m} = -\frac{1}{\tau^2}\partial_\tau \tau^2 \partial_\tau + \frac{1}{\tau^2 \sin^2(\alpha\varphi)}\left(m + \frac{\sin^2(\alpha\varphi)}{2\alpha}\tau^2\right)^2 - \frac{1}{\alpha^2\, \tau^2 \sin(\alpha\varphi)}\partial_\varphi \sin(\alpha\varphi)\partial_\varphi,$$

with

$$\mathcal{R} = \{(\tau,\varphi) \in \mathbb{R}^2,\ \tau > 0,\ \varphi \in (0, \tfrac{1}{2})\},$$

and

$$\mathrm{d}\mu = \tau^2 \sin(\alpha\varphi)\, \mathrm{d}\tau\, \mathrm{d}\varphi.$$

This normal form is also the suitable form to construct quasimodes. Then an integrability argument proves that the eigenfunctions are microlocalized in $m = 0$, i.e. they are axisymmetric. Thus this allows a first reduction of dimension. It remains to notice that the last term in $\mathcal{L}_{\alpha,0,0}$ is penalized by α^{-2} so that the Feshbach-Grushin projection on the groundstate of $-\alpha^{-2}(\sin(\alpha\varphi))^{-1}\partial_\varphi \sin(\alpha\varphi)\partial_\varphi$ (the constant function) acts as an approximation of the identity on the eigenfunctions. Therefore the spectrum of $\mathcal{L}_{\alpha,0,0}$ is described modulo lower order terms by the spectrum of the average of $\mathcal{L}_{\alpha,0}$ with respect to φ which involves the so-called Laguerre operator (radial harmonic oscillator).

1.3.3. *Case* $\beta \in \left[0, \frac{\pi}{2}\right]$. In this case we cannot use the axisymmetry, but we are still able to construct formal series and prove localization estimates of Agmon type. Moreover we notice that the magnetic momentum with respect to θ is strongly penalized by $(\tau^2 \sin^2(\alpha\varphi))^{-1}$ so that, jointly with the localization estimates it is possible to prove that the eigenfunctions are asymptotically independent from θ and we are reduced to the situation $\beta = 0$.

2. Vanishing magnetic fields and boundary

2.1. Why considering vanishing magnetic fields? A motivation is related to the papers of R. Montgomery [111], X-B. Pan and K-H. Kwek [119] and B. Helffer and Y. Kordyukov [68] (see also [74] and [66]) where the authors analyze the spectral influence of the cancellation of the magnetic field in the semiclassical limit. Another motivation appears in the paper [36] where the authors are concerned with the "magnetic waveguides" and inspired by the physical considerations [123, 65] (see also [85]). In any case the case of vanishing magnetic fields can inspire the analysis of non trivial examples of magnetic normal forms, as we will see later.

2.2. Montgomery operator. Without going into the details let us describe the model operator introduced in [111]. Montgomery was concerned by the magnetic Laplacian $(-ih\nabla + \mathbf{A})^2$ on $\mathsf{L}^2(\mathbb{R}^2)$ in the case when the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ vanishes along a smooth curve Γ. Assuming that the magnetic field non degenerately vanishes, he was led to consider the self-adjoint realization on $\mathsf{L}^2(\mathbb{R}^2)$ of:

$$\mathfrak{L} = D_t^2 + (D_s - st)^2.$$

In this case the magnetic field is given by $\beta(s,t) = s$ so that the zero locus of β is the line $s = 0$. Let us write the following change of gauge:

$$\mathfrak{L}^{\mathsf{Mo}} = e^{-i\frac{s^2 t}{2}}\,\mathfrak{L}\,e^{i\frac{s^2 t}{2}} = D_s^2 + \left(D_t + \frac{s^2}{2}\right)^2.$$

The Fourier transform (after changing ζ in $-\zeta$) with respect to t gives the direct integral:

$$\mathfrak{L}^{\mathsf{Mo}} = \int^{\oplus} \mathfrak{L}_\zeta^{[1]}\,\mathrm{d}\zeta, \quad \text{where} \quad \mathfrak{L}_\zeta^{[1]} = D_s^2 + \left(\zeta - \frac{s^2}{2}\right)^2.$$

Note that this family of model operators will be seen as special case of a more general family in Section 3.2. Let us recall a few important properties of the lowest eigenvalue $\nu_1^{[1]}(\zeta)$ of $\mathfrak{L}_\zeta^{[1]}$ (for the proofs, see [119, 67, 80]).

Proposition 2.5. *The following properties hold:*

(1) For all $\zeta \in \mathbb{R}$, $\nu_1^{[1]}(\zeta)$ is simple.
(2) The function $\zeta \mapsto \nu_1^{[1]}(\zeta)$ is analytic.
(3) We have: $\lim_{|\zeta|\to+\infty} \nu_1^{[1]}(\zeta) = +\infty$.
(4) The function $\zeta \mapsto \nu_1^{[1]}(\zeta)$ admits a unique minimum at a point $\zeta_0^{[1]}$ and it is non degenerate.

We have:

$$\mathsf{sp}(\mathfrak{L}) = \mathsf{sp}_{\mathsf{ess}}(\mathfrak{L}) = [\nu_{\mathsf{Mo}}, +\infty)\,, \tag{2.2.1}$$

with $\nu_{\mathsf{Mo}} = \nu_1^{[1]}(\zeta_0^{[1]})$. With a finite element method and Dirichlet condition on the artificial boundary, a upper-bound of the minimum is obtained in [80, Table 1] and the numerical simulations provide $\nu_{\mathsf{Mo}} \simeq 0.5698$ reached for $\zeta_0^{[1]} \simeq 0.3467$ with a discretization step at 10^{-4} for the parameter ζ. This numerical estimate is already mentioned in [111]. In fact we can prove the following lower bound (see [BR13b] for a proof using the Temple inequality).

Proposition 2.6. *We have: $\nu_{\mathsf{Mo}} \geq 0.5$.*

2.3. Generalized Montgomery operators. It turns out that we can generalize the Montgomery operator by allowing an higher order of degeneracy of the magnetic field. Let k be a positive integer. The generalized Montgomery operator of order k is the self-adjoint realization on $\mathbb{R}$ defined by:

$$\mathfrak{L}_\zeta^{[k]} = D_t^2 + \left(\zeta - \frac{t^{k+1}}{k+1}\right)^2 .$$

The following theorem (which generalizes Proposition 2.5) is proved in [54, Theorem 1.3].

Theorem 2.7. *$\zeta \mapsto \nu_1^{[k]}(\zeta)$ admits a unique and non-degenerate minimum at $\zeta = \zeta_0^{[k]}$.*

Notation 2.8. *For real ζ, the lowest eigenvalue of $\mathfrak{L}_\zeta^{[k]}$ is denoted by $\nu_1^{[k]}(\zeta)$ and we denote by $u_\zeta^{[k]}$ the positive and L^2-normalized eigenfunction associated with $\nu_1^{[k]}(\zeta)$. We denote in the same way its holomorphic extension near $\zeta_0^{[k]}$.*

2.4. A broken Montgomery operator.

2.4.1. *Heuristics and motivation.* As mentioned above, the bottom of the spectrum of $\mathfrak{L}$ is essential. This fact is due to the translation invariance along the zero locus of $\mathbf{B}$. This situation reminds what happens in the waveguides framework. Guided by the ideas developed for the waveguides, the papers [**BR13b**] and [**R14a**] aimed at analyzing the effect of breaking the zero locus of $\mathbf{B}$. Introducing the "breaking parameter" $\theta \in (-\pi, \pi]$, we break the invariance of the zero locus in three different ways:

(1) Case with Dirichlet boundary: $\mathfrak{L}_\theta^{\mathsf{Dir}}$. We let $\mathbb{R}_+^2 = \{(s,t) \in \mathbb{R}^2, t > 0\}$ and consider $\mathfrak{L}_\theta^{\mathsf{Dir}}$ the Dirichlet realization, defined as a Friedrichs extension, on $\mathsf{L}^2(\mathbb{R}_+^2)$ of:
$$D_t^2 + \left(D_s + \frac{t^2}{2}\cos\theta - st\sin\theta\right)^2 .$$
(2) Case with Neumann boundary: $\mathfrak{L}_\theta^{\mathsf{Neu}}$. We consider $\mathfrak{L}_\theta^{\mathsf{Neu}}$ the Neumann realization, defined as a Friedrichs extension, on $\mathsf{L}^2(\mathbb{R}_+^2)$ of:
$$D_t^2 + \left(D_s + \frac{t^2}{2}\cos\theta - st\sin\theta\right)^2 .$$
The corresponding magnetic field is $\mathbf{B}(s,t) = t\cos\theta - s\sin\theta$. It cancels along the half-line $t = s\tan\theta$. Note that this model plays a central role in the semiclassical problem when the cancellation line of the magnetic field meets a Neumann boundary (as we can see in [119] and in recent results of my student Miqueu [108]).
(3) Magnetic broken line: $\mathfrak{L}_\theta$. We consider $\mathfrak{L}_\theta$ the Friedrichs extension on $\mathsf{L}^2(\mathbb{R}^2)$ of:
$$D_t^2 + \left(D_s + \mathsf{sgn}(t)\frac{t^2}{2}\cos\theta - st\sin\theta\right)^2 .$$
The corresponding magnetic field is $\beta(s,t) = |t|\cos\theta - s\sin\theta$; it is a continuous function which cancels along the broken line $|t| = s\tan\theta$.

Notation 2.9. *We use the notation $\mathfrak{L}_\theta^\bullet$ where $\bullet$ can be* Dir, Neu *or* $\emptyset$.

2.4.2. *Properties of the spectra.* Let us analyze the dependence of the spectra of $\mathfrak{L}_\theta^\bullet$ on the parameter θ. Denoting by S the axial symmetry $(s,t)\mapsto(-s,t)$, we get:

$$\mathfrak{L}_{-\theta}^\bullet = S\overline{\mathfrak{L}_\theta^\bullet}S,$$

where the line denotes the complex conjugation. Then, we notice that $\mathfrak{L}_\theta^\bullet$ and $\overline{\mathfrak{L}_\theta^\bullet}$ are isospectral. Therefore, the analysis is reduced to $\theta\in[0,\pi)$. Moreover, we get:

$$S\mathfrak{L}_\theta^\bullet S = \mathfrak{L}_{\pi-\theta}^\bullet.$$

The study is reduced to $\theta\in\left[0,\frac{\pi}{2}\right]$. We observe that at $\theta=0$ and $\theta=\frac{\pi}{2}$ the domain of $\mathfrak{L}_\theta^\bullet$ is not continuous.

Lemma 2.10. *The family $(\mathfrak{L}_\theta^\bullet)_{\theta\in\left(0,\frac{\pi}{2}\right)}$ is analytic of type (A).*

The following proposition states that the infimum of the essential spectrum is the same for $\mathfrak{L}_\theta^{\mathsf{Dir}}$, $\mathfrak{L}_\theta^{\mathsf{Neu}}$ and $\mathfrak{L}_\theta$.

Proposition 2.11. *For $\theta\in\left(0,\frac{\pi}{2}\right)$, we have $\inf \mathsf{sp}_{\mathsf{ess}}(\mathfrak{L}_\theta^\bullet)=\nu_{\mathsf{Mo}}$.*

In the Dirichlet case, the spectrum is essential:

Proposition 2.12. *For all $\theta\in\left(0,\frac{\pi}{2}\right)$, we have $\mathsf{sp}(\mathfrak{L}_\theta^{\mathsf{Dir}})=[\nu_{\mathsf{Mo}},+\infty)$.*

From now on we assume that $\bullet=\mathsf{Neu},\emptyset$.

Notation 2.13. *Let us denote by $\lambda_n^\bullet(\theta)$ the n-th number in the sense of the Rayleigh variational formula for $\mathfrak{L}_\theta^\bullet$.*

The two following propositions are Agmon type estimates and give an exponential decay of the eigenfunctions (a proof is given in [**BR13b**]). $\mathbb{R}^2_\bullet$ denotes $\mathbb{R}^2_+$, $\mathbb{R}^2$ when $\bullet=\mathsf{Neu},\emptyset$ respectively. The first decay is proved with respect to the variable t.

Proposition 2.14. *There exist $\varepsilon_0, C>0$ such that for all $\theta\in\left(0,\frac{\pi}{2}\right)$ and all eigenpair (λ,ψ) of $\mathfrak{L}_\theta^\bullet$ such that $\lambda<\nu_{\mathsf{Mo}}$, we have:*

$$\int_{\mathbb{R}^2_\bullet} e^{2\varepsilon_0|t|\sqrt{\nu_{\mathsf{Mo}}-\lambda}}|\psi|^2\,\mathrm{d}s\,\mathrm{d}t\leq C(\nu_{\mathsf{Mo}}-\lambda)^{-1}\|\psi\|^2.$$

The second decay is related to the (semiclassical) variable s and is not optimal at all when θ goes to zero (see [**R14a**] ; we will again meet this non optimality in our magnetic Born-Oppenheimer theory).

Proposition 2.15. *There exist $\varepsilon_0, C>0$ such that for all $\theta\in\left(0,\frac{\pi}{2}\right)$ and all eigenpair (λ,ψ) of $\mathfrak{L}_\theta^\bullet$ such that $\lambda<\nu_{\mathsf{Mo}}$, we have:*

$$\int_{\mathbb{R}^2_\bullet} e^{2\varepsilon_0|s|\sin\theta\sqrt{\nu_{\mathsf{Mo}}-\lambda}}|\psi|^2\,\mathrm{d}s\,\mathrm{d}t\leq C(\nu_{\mathsf{Mo}}-\lambda)^{-1}\|\psi\|^2.$$

The following proposition (the proof of which can be found in [**119**, Lemma 5.2]) states that $\mathfrak{L}_\theta^{\mathsf{Neu}}$ admits an eigenvalue below its essential spectrum when $\theta\in\left(0,\frac{\pi}{2}\right]$.

Proposition 2.16. *For all* $\theta \in \left(0, \frac{\pi}{2}\right]$, $\lambda_1^{\mathsf{Neu}}(\theta) < \nu_{\mathsf{Mo}}$.

Remark 2.17. *The situation seems to be different for* $\mathfrak{L}_\theta$. *According to numerical simulations with a finite elements method, there exists* $\theta_0 \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ *such that* $\lambda_1(\theta) < \nu_{\mathsf{Mo}}$ *for all* $\theta \in (0, \theta_0)$ *and* $\lambda_1(\theta) = \nu_{\mathsf{Mo}}$ *for all* $\theta \in \left[\theta_0, \frac{\pi}{2}\right)$.

2.5. Singular limit $\theta \to 0$.

2.5.1. *Renormalization.* Thanks to Proposition 2.16, one knows that breaking the invariance of the zero locus of the magnetic field with a Neumann boundary creates a bound state. We also would like to tackle this question for $\mathfrak{L}_\theta$ and in any case to estimate more quantitatively this effect: this was the specific purpose of [**R14a**]. A way to do this is to consider the limit $\theta \to 0$ which reveals new model operators. First, we perform a scaling:

$$s = (\cos\theta)^{-1/3}\hat{s}, \quad t = (\cos\theta)^{-1/3}\hat{t}. \tag{2.2.2}$$

The operator $\mathfrak{L}_\theta^\bullet$ is thus unitarily equivalent to $(\cos\theta)^{2/3}\hat{\mathfrak{L}}^\bullet_{\tan\theta}$, where the expression of $\hat{\mathfrak{L}}^\bullet_{\tan\theta}$ is given by:

$$D_{\hat{t}}^2 + \left(D_{\hat{s}} + \mathsf{sgn}(\hat{t})\frac{\hat{t}^2}{2} - \hat{s}\hat{t}\tan\theta\right)^2.$$

Notation 2.18. *We let* $\varepsilon = \tan\theta$.

For $(x, \xi) \in \mathbb{R}^2$ and $\varepsilon > 0$, we introduce the unitary transform:

$$V_{\varepsilon,x,\xi}\psi(\hat{s},\hat{t}) = e^{-i\xi\hat{s}}\psi\left(\hat{s} - \frac{x}{\varepsilon}, \hat{t}\right),$$

and the conjugate operator:

$$\hat{\mathfrak{L}}^\bullet_{\varepsilon,x,\xi} = V^{-1}_{\varepsilon,x,\xi}\hat{\mathfrak{L}}^\bullet_\varepsilon V_{\varepsilon,x,\xi}.$$

Its expression is given by:

$$\hat{\mathfrak{L}}^\bullet_{\varepsilon,x,\xi} = D_{\hat{t}}^2 + \left(-\xi - x\hat{t} + \mathsf{sgn}(\hat{t})\frac{\hat{t}^2}{2} + D_{\hat{s}} - \varepsilon\hat{s}\hat{t}\right)^2. \tag{2.2.3}$$

Let us introduce the new variables:

$$\hat{s} = \varepsilon^{-1/2}\sigma, \quad \hat{t} = \tau \tag{2.2.4}$$

Therefore $\hat{\mathfrak{L}}^\bullet_{\varepsilon,x,\xi}$ is unitarily equivalent to $\mathfrak{L}^\bullet_{\varepsilon,x,\xi}$ whose expression is given by:

$$\mathfrak{L}^\bullet_{\varepsilon,x,\xi} = D_\tau^2 + \left(-\xi - x\tau + \mathsf{sgn}(\tau)\frac{\tau^2}{2} + \varepsilon^{1/2}D_\sigma - \varepsilon^{1/2}\sigma\tau\right)^2. \tag{2.2.5}$$

2.5.2. *New model operators.* By taking formally $\varepsilon = 0$ in (2.2.5) we are led to two families of one dimensional operators on $\mathsf{L}^2(\mathbb{R}^2_\bullet)$ with two parameters $(x, \xi) \in \mathbb{R}^2$:

$$\mathcal{M}^\bullet_{x,\xi} = D_\tau^2 + \left(-\xi - x\tau + \mathsf{sgn}(\tau)\frac{\tau^2}{2}\right)^2.$$

These operators have compact resolvents and are analytic families with respect to the variables $(x, \xi) \in \mathbb{R}^2$.

Notation 2.19. *We denote by* $\mu_n^{\bullet}(x, \xi)$ *the* n*-th eigenvalue of* $\mathcal{M}_{x,\xi}^{\bullet}$.

Roughly speaking $\mathcal{M}_{x,\xi}^{\bullet}$ is the operator valued symbol of (2.2.5), so that we expect that the behavior of the so-called "band function" $(x, \xi) \mapsto \mu_1^{\bullet}(x, \xi)$ determines the structure of the low lying spectrum of $\mathfrak{M}_{\varepsilon,x,\xi}^{\bullet}$ in the limit $\varepsilon \to 0$. The following two theorems (proved in [BR13b]) state that the band functions admit a minimum.

Theorem 2.20. *The function* $\mathbb{R} \times \mathbb{R} \ni (x, \xi) \mapsto \mu_1^{\mathsf{Neu}}(x, \xi)$ *admits a minimum denoted by* $\underline{\mu}_1^{\mathsf{Neu}}$. *Moreover we have:*

$$\liminf_{|x|+|\xi| \to +\infty} \mu_1^{\mathsf{Neu}}(x, \xi) \geq \nu_{\mathsf{Mo}} > \min_{(x,\xi) \in \mathbb{R}^2} \mu_1^{\mathsf{Neu}}(x, \xi) = \underline{\mu}_1^{\mathsf{Neu}}.$$

Theorem 2.21. *The function* $\mathbb{R} \times \mathbb{R} \ni (x, \xi) \mapsto \mu_1(x, \xi)$ *admits a minimum denoted by* $\underline{\mu}_1$. *Moreover we have:*

$$\liminf_{|x|+|\xi| \to +\infty} \mu_1(x, \xi) \geq \nu_{\mathsf{Mo}} > \min_{(x,\xi) \in \mathbb{R}^2} \mu_1(x, \xi) = \underline{\mu}_1.$$

Numerical experiments lead to the following conjecture.

Conjecture 2.22. *The minimum* $\underline{\mu}_1^{\bullet}$ *is unique (and attained at* (x_0, ξ_0)*) and non-degenerate.*

Under Conjecture 2.22, it is possible to prove complete asymptotic expansions of the first eigenvalues of $\mathfrak{L}_\theta$ (see [R14a]). In order to state this result, we introduce

$$\mathcal{H} = \frac{\partial_\xi^2 \mu_1(x_0, \xi_0)}{2} D_\sigma^2 - \frac{\partial_\xi \partial_\alpha \mu_1(x_0, \xi_0)}{2} \sigma D_\sigma - \frac{\partial_\xi \partial_\alpha \mu_1(x_0, \xi_0)}{2} D_\sigma \sigma + \frac{\partial_\alpha^2 \mu_1(x_0, \xi_0)}{2} \sigma^2. \tag{2.2.6}$$

Theorem 2.23. *We assume that Conjecture 2.22 is true. For all* $n \geq 1$, *there exists a sequence* $(\delta_j^n)_{j \geq 0}$ *such that the* n*-th eigenvalue of* $\mathfrak{L}_\theta$ *exists and satisfies*

$$\lambda_n(\theta) \underset{\theta \to 0}{\sim} \sum_{j \geq 0} \delta_j^n \theta^{j/2},$$

with:

$$\delta_0^n = \mu_0, \quad \delta_1^n = 0$$

where μ_0 *is the infimum of the band function* μ_1 *and* δ_2^n *is the* n*-th eigenvalue of* $\mathcal{H}$.

Remark 2.24. *Theorem 2.23 implies that the lowest eigenvalues become simple when* θ *is small enough so that we get an approximation at any order of the corresponding eigenfunctions by some formal power series which behave like Hermite's functions with respect to* $\sigma = (\tan\theta)^{1/2}\hat{s}$ *at the main order. These eigenfunctions are microlocalized near* $(\sigma, D_\sigma) = (x_0, \xi_0)$ *and have the same behavior as the computed eigenfunctions displayed on Figures 2 and 3. Note that, in order to prove this theorem, we have used in* [R14a] *a coherent states decomposition which seem to be an unusual tool to study the low lying spectrum of the semiclassical Laplacian. Implementing the idea was also a motivation for* *this work ; this was the first step towards a more general theory: Theorem 2.23 can be proved by using the magnetic Born-Oppenheimer approximation (see Section 3).*

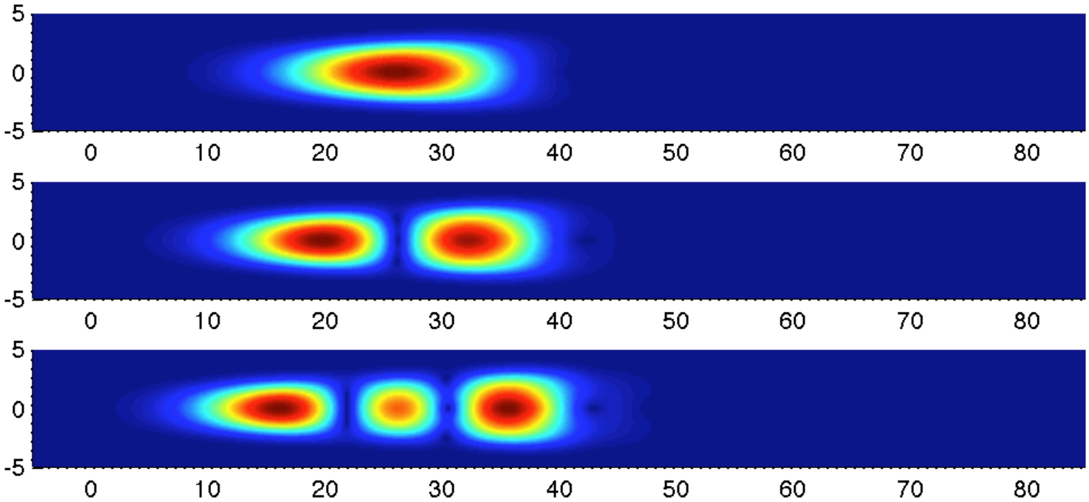


FIGURE 2. Modulus of the first three eigenfunctions of $\hat{\mathfrak{L}}_{\tan\theta}$ when $\theta = \frac{\pi}{100}$.

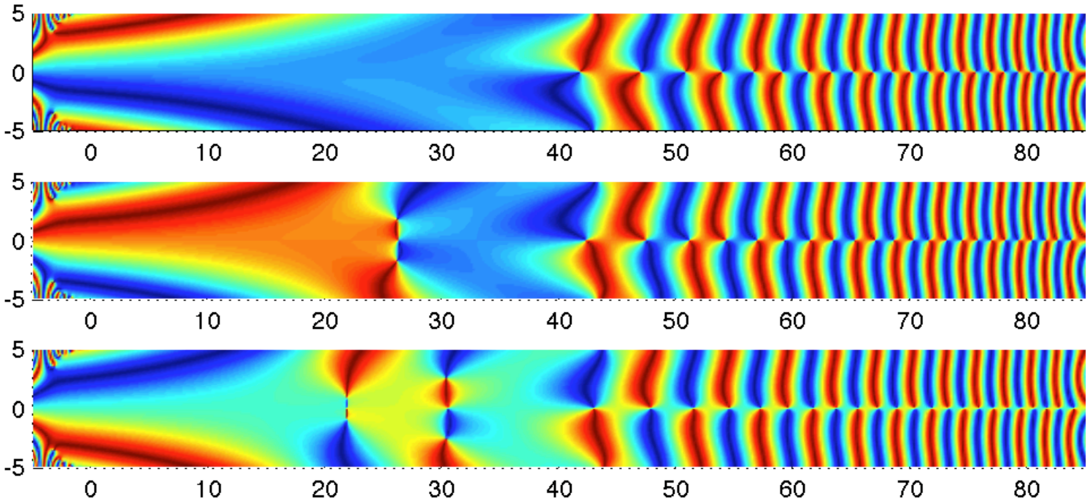


FIGURE 3. Phase of the first three eigenfunctions of $\hat{\mathfrak{L}}_{\tan\theta}$ when $\theta = \frac{\pi}{100}$.

3. Magnetic Born-Oppenheimer approximation

The results discussed below are obtained in collaboration with V. Bonnaillie-Noël and F. Hérau (see [BHR14]) and generalize the strategy used in [R14a]. This section is devoted to the analysis of the operator on $\mathsf{L}^2(\mathbb{R}^m_s \times \mathbb{R}^n_t, \mathrm{d}s\,\mathrm{d}t)$:

$$\mathfrak{L}_h = (-ih\nabla_s + A_1(s,t))^2 + (-i\nabla_t + A_2(s,t))^2. \tag{2.3.1}$$

Note that (2.2.3) can easily be put in this form. For simplicity's sake we will assume that A_1 and A_2 are analytic. We would like to describe the lowest eigenvalues of this operator in the limit $h \to 0$ under elementary confining assumptions. The problem of considering partial semiclassical problems appears for instance in the context of [103, 90] where the main issue is to approximate the eigenvalues and eigenfunctions of operators in the form:

$$-h^2\Delta_s - \Delta_t + V(s,t). \tag{2.3.2}$$

The main idea, due to Born and Oppenheimer in [18], is to replace, for fixed s, the operator $-\Delta_t + V(s,t)$ by its eigenvalues $\mu_k(s)$. Then we are led to consider for instance the reduced operator (called Born-Oppenheimer approximation):

$$-h^2\Delta_s + \mu_1(s)$$

and to apply the semiclassical techniques *à la* Helffer-Sjöstrand [81, 82] to analyze in particular the tunnel effect when the potential μ_1 admits symmetries. The main point it to make the reduction of dimension rigorous. Note that we have always the following lower bound:

$$-h^2\Delta_s - \Delta_t + V(s,t) \geq -h^2\Delta_s + \mu_1(s), \tag{2.3.3}$$

which involves accurate estimates of Agmon with respect to s.

3.1. Electric Born-Oppenheimer approximation and low lying spectrum.

Before dealing with the so-called Born-Oppenheimer approximation in presence of magnetic fields, we shall recall the philosophy in a simplified electric case.

3.1.1. *Electric result.* Let us explain the question in which we are interested. We shall study operators in $\mathsf{L}^2(\mathbb{R} \times \Omega)$ (with $\Omega \subset \mathbb{R}^n$) in the form:

$$\mathfrak{H}_h = h^2 D_s^2 + \mathcal{V}(s),$$

where $\mathcal{V}(s) = -\Delta_t + P(t,s)$ is a family of semi-bounded self-adjoint operators, analytic of type (B), with P polynomial for simplicity. We will denote by $\mathfrak{Q}_h$ the corresponding quadratic form.

We want to analyze the low lying eigenvalues of this operator. We will assume that the lowest eigenvalue $\nu(s)$ of $\mathcal{V}(s)$ (which is simple) admits, as a function of s, a unique and non degenerate minimum at s_0.

We now discuss the heuristics. We hope that $\mathfrak{H}_h$ can be described by its "Born-Oppenheimer" approximation:

$$\mathfrak{H}_h^{\mathsf{BO}} = h^2 D_s^2 + \mu(s),$$

which is an electric Laplacian in dimension one. Then, we guess that $\mathfrak{H}_h^{\mathsf{BO}}$ is well approximated by its Taylor expansion:

$$h^2 D_s^2 + \mu(s_0) + \frac{\nu''(s_0)}{2}(s-s_0)^2.$$

In fact this heuristics can be made rigorous.

Assumption 2.25. *Let us assume that* $\liminf_{s\to\pm\infty} \nu(s) > \nu(s_0)$ *and that*

$$\inf_s \mathsf{sp}_{\mathsf{ess}}(\mathcal{V}(s)) > \nu(s_0).$$

Theorem 2.26. *Let us assume that* $\nu(s)$ *admits a unique and non degenerate minimum at* s_0 *and that Assumption 2.25 is satisfied. Then the* n*-th eigenvalue of* $\mathfrak{H}_h$ *has the expansion*

$$\lambda_n(h) = \nu(s_0) + h(2n-1)\left(\frac{\nu''(s_0)}{2}\right)^{1/2} + o(h).$$

3.1.2. *A non example: the broken* δ*-interactions.* In the last theorem we were only interested in the low lying spectrum. It turns out that the so-called Born-Oppenheimer reduction is a slightly more general procedure (see [**103**, **90**]) which provides in general an effective Hamiltonian which describes the spectrum below some fixed energy level (and allows for instance to estimate the counting function). With the example of broken δ-interactions, the standard technique needs to be adapted due to the singularity of the δ interaction (one may consult [**46**, **47**, **21**, **45**] for perspectives and motivation). The results presented below are obtained in collaboration with V. Duchêne in [**DuR14**]. Let us consider $\mathfrak{H}_h$ the Friedrichs extension (see [**20**]) of the rescaled quadratic form:

$$\mathfrak{Q}_h(\psi) = \int_{\mathbb{R}^2} h^2|\partial_x\psi|^2 + |\partial_y\psi|^2 \,\mathrm{d}x\,\mathrm{d}y - \int_{\mathbb{R}} |\psi(|s|,s)|^2\,\mathrm{d}s, \quad \forall\psi\in H^1(\mathbb{R}^2). \tag{2.3.4}$$

Formally we may write

$$\mathfrak{H}_h = -h^2\partial_x^2 - \partial_y^2 - \delta_{\Sigma_{\frac{\pi}{4}}}, \tag{2.3.5}$$

where

$$\Sigma_{\frac{\pi}{4}} = \{(|s|,s), \quad s\in\mathbb{R}\}.$$

In particular, we notice that:

$$\mathsf{sp}_{\mathsf{ess}}(\mathfrak{H}_h) = \left[-\frac{1}{4(1+h^2)}, +\infty\right).$$

Let us introduce some notation.

Notation 2.27. *We denote by* $W : [-e^{-1},+\infty) \to [-1,+\infty)$ *the Lambert function defined as the inverse of* $[-1,+\infty) \ni w \mapsto we^w \in [-e^{-1},+\infty)$.

Notation 2.28. *Given $\mathfrak{H}$ a semi-bounded self-adjoint operator and $a < \inf \mathsf{sp}_{\mathsf{ess}}(\mathfrak{H})$, we denote*

$$\mathsf{N}(\mathfrak{H}, a) \;=\; \#\{\lambda \in \mathsf{sp}(\mathfrak{H}) \;:\; \lambda \leq a\} < +\infty.$$

The eigenvalues are counted with multiplicity.

The following theorem provides the asymptotics of the number of bound states.

Theorem 2.29. *There exists $M_0 > 0$ such that for all $C(h) \geq M_0 h$ with $C(h) \underset{h\to 0}{\to} C_0 \geq 0$:*

$$\mathsf{N}\left(\mathfrak{H}_h, -\frac{1}{4} - C(h)\right) \underset{h\to 0}{\sim} \frac{1}{\pi h} \int_{x=0}^{+\infty} \sqrt{-\frac{1}{4} - C_0 + \left(\frac{1}{2} + \frac{1}{2x} W(xe^{-x})\right)^2}\, \mathrm{d}x.$$

Remark 2.30. *It is important to notice that in the above result, we estimate the counting function below a potentially moving (w.r.t. h) threshold. In particular, the distance between $-\frac{1}{4} - C(h)$ and the bottom of the essential spectrum is allowed to vanish in the semiclassical limit. Therefore our statement is slightly unusual as customary results would typically concern $\mathsf{N}(\mathfrak{H}_h, E)$ with E fixed and satisfying $E < -\frac{1}{4}$, so as to insure a fixed security distance to the bottom of the essential spectrum (see for instance the related works* [**6**, **112**]*).*

The next theorem is the analogous of Theorem 2.26.

Theorem 2.31. *For all $n \geq 1$, we have:*

$$\lambda_n(h) \underset{h\to 0}{=} -1 + 2^{2/3} z_{\mathsf{Ai}^{\mathsf{rev}}}(n) h^{2/3} + O(h),$$

where $z_{\mathsf{Ai}^{\mathsf{rev}}}(n)$ is the n-th zero of the reversed Airy function.

3.2. Magnetic case. We would like to understand the analogy between (2.3.1) and (2.3.2). In particular even the formal dimensional reduction does not seem to be as clear as in the electric case. Let us write the operator valued symbol of $\mathfrak{L}_h$. For $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$, we introduce the electro-magnetic Laplacian acting on $\mathsf{L}^2(\mathbb{R}^n, \mathrm{d}t)$:

$$\mathcal{M}_{x,\xi} = (-i\nabla_t + A_2(x,t))^2 + (\xi + A_1(x,t))^2.$$

Denoting by $\mu_1(x,\xi) = \mu(x,\xi)$ its lowest eigenvalue we would like to replace $\mathfrak{L}_h$ by the m-dimensional pseudo-differential operator:

$$\mu(s, -ih\nabla_s).$$

Under different assumptions, such reductions are considered in [**106**, Theorem 2.1 and remark thereafter] where it is suggested that the spectrum of $\mathfrak{L}_h$ could be completely determined by an effective Hamiltonian (a matrix of pseudo-differential operators) whose principal symbol can be described thanks to the spectral invariants of the operator valued symbol of $\mathfrak{L}_h$. For the present situation the low lying spectrum of $\mathfrak{L}_h$ could be described by the one of $\mu(s, hD_s)$ modulo $\mathcal{O}(h)$ and we will see that, under generic assumptions,

$\mathcal{O}(h)$ is precisely the order of the spectral gap between the first eigenvalues in the simple well case.

3.2.1. *Eigenvalue asymptotics in the magnetic Born-Oppenheimer approximation.* We work under the following assumptions. The first assumption essentially states that the lowest eigenvalue of the operator symbol of $\mathfrak{L}_h$ admits a unique and non-degenerate minimum.

Assumption 2.32.
- *The family $(\mathcal{M}_{x,\xi})_{(x,\xi)\in\mathbb{R}^m\times\mathbb{R}^m}$ is analytic of type (B) in the sense of Kato* [89, Chapter VII].
- *For all $(x,\xi)\in\mathbb{R}^m\times\mathbb{R}^m$, the bottom of the spectrum of $\mathcal{M}_{x,\xi}$ is a simple eigenvalue denoted by $\mu(x,\xi)$ (in particular it is an analytic function) and associated with a L^2-normalized eigenfunction $u_{x,\xi}\in\mathcal{S}(\mathbb{R}^n)$ which also analytically depends on (x,ξ).*
- *The function μ admits a unique and non degenerate minimum μ_0 at point denoted by (x_0,ξ_0) and such that $\liminf_{|x|+|\xi|\to+\infty}\mu(x,\xi)>\mu_0$.*
- *The family $(\mathcal{M}_{x,\xi})_{(x,\xi)\in\mathbb{R}^m\times\mathbb{R}^m}$ can be analytically extended in a complex neighborhood of (x_0,ξ_0).*

Assumption 2.33. *Under Assumption 2.32, let us denote by $\mathsf{Hess}\,\mu(x_0,\xi_0)$ the Hessian matrix of μ at (x_0,ξ_0). We assume that the spectrum of $\mathsf{Hess}\,\mu(x_0,\xi_0)(\sigma,D_\sigma)$ is simple.*

The next assumption is a spectral confinement.

Assumption 2.34. *For $R\geq 0$, we let $\Omega_R=\mathbb{R}^{m+n}\setminus\overline{B(0,R)}$. We denote by $\mathfrak{L}_h^{\mathsf{Dir},\Omega_R}$ the Dirichlet realization on Ω_R of $(-i\nabla_t+A_2(s,t))^2+(-ih\nabla_s+A_1(s,t))^2$. We assume that there exist $R_0\geq 0$, $h_0>0$ and $\mu_0^*>\mu_0$ such that for all $h\in(0,h_0)$:*

$$\lambda_1^{\mathsf{Dir},\Omega_{R_0}}(h)\geq\mu_0^*.$$

Remark 2.35. *In particular, due to the monotonicity of the Dirichlet realization with respect to the domain, Assumption 2.34 implies that there exist $R_0>0$ and $h_0>0$ such that for all $R\geq R_0$ and $h\in(0,h_0)$:*

$$\lambda_1^{\mathsf{Dir},\Omega_R}(h)\geq\lambda_1^{\mathsf{Dir},\Omega_{R_0}}(h)\geq\mu_0^*.$$

By using the Persson's theorem, we have the following proposition.

Proposition 2.36. *Let us assume Assumption 2.34. There exists $h_0>0$ such that for all $h\in(0,h_0)$:*

$$\inf\mathsf{sp}_{\mathsf{ess}}(\mathfrak{L}_h)\geq\mu_0^*.$$

We can now state the theorem concerning the spectral asymptotics.

Theorem 2.37. *We assume that A_1 and A_2 are polynomials. Let us assume Assumptions 2.32, 2.33 and 2.34. For all $n\geq 1$, there exist a sequence $(\gamma_{j,n})_{j\geq 0}$ and $h_0>0$ such that* *for all $h \in (0, h_0)$ the n-th eigenvalue of $\mathfrak{L}_h$ exists and satifies:*

$$\lambda_n(h) \underset{h\to 0}{\sim} \sum_{j\geq 0} \gamma_{j,n} h^{j/2},$$

where $\gamma_{0,n} = \mu_0, \quad \gamma_{1,n} = 0$ and $\mu_{2,n}$ is the n-th eigenvalue of $\frac{1}{2}\mathsf{Hess}_{x_0,\xi_0}\, \mu(\sigma, D_\sigma)$.

3.2.2. *Using the coherent states to prove Theorem 2.37.* Let us very roughly explain the structure of the proof of Theorem 2.37. We use the following rescaling

$$s = x_0 + h^{1/2}\sigma, \qquad t = \tau, \tag{2.3.6}$$

and a gauge transform $e^{i\xi_0\sigma/h^{1/2}}$, so that $\mathfrak{L}_h$ becomes

$$\mathcal{L}_h = (-i\nabla_\tau + A_2(x_0 + h^{1/2}\sigma, \tau))^2 + (\xi_0 - ih^{1/2}\nabla_\sigma + A_1(x_0 + h^{1/2}\sigma, \tau))^2. \tag{2.3.7}$$

The first step in the proof of Theorem 2.37 is, as usual, a construction of quasimodes (which behave like the Hermite functions with respect to σ). Here it involves generalizations of the Feynman-Hellmann formulas (which are consequences of the Kato theory) jointly with the classical Fredholm alternative. The second step is more difficult and involves a microlocal analysis of the eigenfunctions (which has to be done to prove that the constructed quasimodes are actually approximations of the eigenfunctions). It turns out that the coherent states representation is flexible enough to succeed. Let us recall the formalism of coherent states (see for instance [50] and [28]) to give the flavor of the proof. We define

$$g_0(\sigma) = \pi^{-m/4} e^{-|\sigma|^2/2},$$

and the usual creation and annihilation operators

$$\mathfrak{a}_j = \frac{1}{\sqrt{2}}(\sigma_j + \partial_{\sigma_j}), \qquad \mathfrak{a}_j^* = \frac{1}{\sqrt{2}}(\sigma_j - \partial_{\sigma_j}),$$

which satisfy the commutator relations

$$[\mathfrak{a}_j, \mathfrak{a}_j^*] = 1, \qquad [\mathfrak{a}_j, \mathfrak{a}_k^*] = 0 \quad \text{if } k \neq j.$$

We notice that

$$\sigma_j = \frac{1}{\sqrt{2}}(\mathfrak{a}_j + \mathfrak{a}_j^*), \qquad \partial_{\sigma_j} = \frac{1}{\sqrt{2}}(\mathfrak{a}_j - \mathfrak{a}_j^*), \qquad \mathfrak{a}_j\mathfrak{a}_j^* = \frac{1}{2}(D_{\sigma_j}^2 + \sigma_j^2 + 1).$$

For $(u, p) \in \mathbb{R}^m \times \mathbb{R}^m$, we introduce the coherent state

$$f_{u,p}(\sigma) = e^{ip\cdot\sigma} g_0(\sigma - u),$$

and the associated projection, defined for $\psi \in \mathsf{L}^2(\mathbb{R}^m \times \mathbb{R}^n)$ by

$$\Pi_{u,p}\psi = \langle \psi, f_{u,p}\rangle_{\mathsf{L}^2(\mathbb{R}^m, \mathrm{d}\sigma)} f_{u,p} = \psi_{u,p} f_{u,p},$$

which satisfies

$$\psi = \int_{\mathbb{R}^{2m}} \Pi_{u,p}\psi \,\mathrm{d}u\,\mathrm{d}p,$$

and the Parseval formula

$$\|\psi\|^2 = \int_{\mathbb{R}^n}\int_{\mathbb{R}^{2m}} |\psi_{u,p}|^2 \,\mathrm{d}u\,\mathrm{d}p\,\mathrm{d}\tau.$$

We recall that

$$\mathfrak{a}_j f_{u,p} = \frac{u_j + ip_j}{\sqrt{2}} f_{u,p}$$

and

$$(\mathfrak{a}_j)^\ell (\mathfrak{a}_k^*)^q \psi = \int_{\mathbb{R}^{2m}} \left(\frac{u_j+ip_j}{\sqrt{2}}\right)^\ell \left(\frac{u_k - ip_k}{\sqrt{2}}\right)^q \Pi_{u,p}\psi \,\mathrm{d}u\,\mathrm{d}p.$$

We have

$$\mathcal{L}_h = \mathcal{L}_0 + h^{1/2}\mathcal{L}_1 + h\mathcal{L}_2 + \ldots + h^{M/2}\mathcal{L}_M.$$

If we write the Wick ordered operator, we get

$$\mathcal{L}_h = \underbrace{\mathcal{L}_0 + h^{1/2}\mathcal{L}_1 + h\mathcal{L}_2^{\mathsf{W}} + \ldots + (h^{1/2})^M \mathcal{L}_M^{\mathsf{W}}}_{\mathcal{L}_h^{\mathsf{W}}} + \underbrace{hR_2 + \ldots + (h^{1/2})^M R_M}_{\mathcal{R}_h}, \tag{2.3.8}$$

where the R_j are the remainders in the (anti-)Wick ordering and satisfy, for $j \geq 2$,

$$h^{j/2}R_j = h^{j/2}\mathcal{O}_{j-2}(\sigma, D_\sigma), \tag{2.3.9}$$

where the notation $\mathcal{O}_j(\sigma, D_\sigma)$ stands for a polynomial operator with total degree in (σ, D_σ) less than j. We recall that

$$\mathcal{L}_h^{\mathsf{W}} = \int_{\mathbb{R}^{2m}} \mathcal{M}_{x_0 + h^{1/2}u, \xi_0 + h^{1/2}p} \,\mathrm{d}u\,\mathrm{d}p.$$

Then, the microlocal analysis of the eigenfunctions can start. Without entering into the details, the main idea is to prove polynomial (in (u,p)) weighted estimates in the phase phase by using the following elementary "microlocalization" lemma (which we proved in [**R14a**]) with $\mathfrak{A}$ a polynomial in $\mathfrak{a}_k$ and $\mathfrak{a}_k^*$.

Lemma 2.38 ("Localization" of P^2 with respect to $\mathfrak{A}$)**.** *Let H be a Hilbert space and P and $\mathfrak{A}$ be two unbounded operators defined on a domain $\mathsf{D} \subset \mathsf{H}$. We assume that P is symmetric and that $P(\mathsf{D}) \subset \mathsf{D}$, $\mathfrak{A}(\mathsf{D}) \subset \mathsf{D}$, $\mathfrak{A}^*(\mathsf{D}) \subset \mathsf{D}$. Then, for $\psi \in \mathsf{D}$, we have*

$$\begin{aligned} \mathsf{Re}\,\langle P^2\psi, \mathfrak{A}\mathfrak{A}^*\psi\rangle = \|P(\mathfrak{A}^*\psi)\|^2 - \|[\mathfrak{A}^*, P]\psi\|^2 + \mathsf{Re}\,\langle P\psi, [[P,\mathfrak{A}],\mathfrak{A}^*]\psi\rangle \\ + \mathsf{Re}\left(\langle P\psi, \mathfrak{A}^*[P,\mathfrak{A}]\psi\rangle - \overline{\langle P\psi, \mathfrak{A}[P,\mathfrak{A}^*]\psi\rangle}\right). \end{aligned} \tag{2.3.10}$$

The obtained estimates, which tell that the eigenfunctions are bounded in σ and D_σ (these bounds are better than the one provided by the naive estimates of Agmon), are then enough to implement a dimensional reduction (to the effective harmonic oscillator) in the Grushin spirit.

3.2.3. *A family of examples.* In order to make our Assumptions 2.32, 2.33 and 2.34 more concrete, let us provide a family of examples in dimension two which is related to [**80**] and the more recent result by Fournais and Persson [**54**]. Our examples are strongly connected with [**68**, Conjecture 1.1 and below].

For $k \in \mathbb{N} \setminus \{0\}$, we consider the operator the following magnetic Laplacian on $\mathsf{L}^2(\mathbb{R}^2, \mathrm{d}x\,\mathrm{d}s)$:

$$\mathfrak{L}_{h,\mathbf{A}^{[k]}} = h^2 D_{\mathsf{t}}^2 + \left(hD_{\mathsf{s}} - \gamma(\mathsf{s})\frac{\mathsf{t}^{k+1}}{k+1}\right)^2,$$

where γ is analytic. Let us assume that either γ is polynomial and admits a unique minimum $\gamma_0 > 0$ at $s_0 = 0$ which is non degenerate, or γ is analytic and such that $\liminf_{x\to\pm\infty} \gamma = \gamma_\infty \in (\gamma_0, +\infty)$.

Let us perform the rescaling:

$$\mathsf{s} = s, \quad \mathsf{t} = h^{\frac{1}{1+k}} t.$$

The operator becomes:

$$h^{\frac{2k+2}{k+2}}\left(D_t^2 + \left(h^{\frac{1}{k+2}} D_s - \gamma(s)\frac{t^{k+1}}{k+1}\right)^2\right).$$

and the investigation is reduced to the one of:

$$\mathfrak{L}_h^{\mathsf{vf},[k]} = D_t^2 + \left(h^{\frac{1}{k+2}} D_s - \gamma(s)\frac{t^{k+1}}{k+1}\right)^2. \tag{2.3.11}$$

Let us verify Assumption 2.32. The $h^{\frac{1}{k+2}}$-symbol of $\mathfrak{L}_h^{[k]}$ with respect to s is:

$$\mathcal{M}_{x,\xi}^{[k]} = D_t^2 + \left(\xi - \gamma(x)\frac{t^{k+1}}{k+1}\right)^2.$$

The lowest eigenvalue of $\mathcal{M}_{x,\xi}^{[k]}$, denoted by $\mu^{[k]}(x,\xi)$, satisfies:

$$\mu^{[k]}(x,\xi) = (\gamma(x))^{\frac{2}{k+2}} \nu_1^{[k]}\left((\gamma(x))^{-\frac{1}{k+2}}\xi\right),$$

where $\nu_1^{[k]}(\zeta)$ denotes the first eigenvalue of:

$$\mathfrak{L}_\zeta^{[k]} = D_t^2 + \left(\zeta - \frac{t^{k+1}}{k+1}\right)^2.$$

We recall the non trivial fact that $\zeta \mapsto \nu_1^{[k]}(\zeta)$ admits a unique and non-degenerate minimum at $\zeta = \zeta_0^{[k]}$ (see Theorem 2.7). Therefore Assumption 2.32 is satisfied. This is delicate to verify Assumption 2.34 and this relies on a basic normal form procedure that we will use for our magnetic WKB constructions.

3.3. The magnetic WKB expansions.

3.3.1. *WKB analysis and estimates of Agmon.* As we explained in Chapter 1, Section 3.2.1, in many papers about asymptotic expansions of the magnetic eigenfunctions, one of the methods consists in using a formal power series expansion. It turns out that these constructions are never in the famous WKB form, but in a weaker and somehow more flexible one. When there is an additional electric potential, the WKB expansions are possible as we can see in [83] and [107]. The reason for which we would like to have a WKB description of the eigenfunctions is to get a precise estimate of the magnetic tunnel effect in the case of symmetries. Until now, such estimates are only investigated in two dimensional corner domains in [11] and [12] for the numerical counterpart. It turns out that the crucial point to get an accurate estimate of the exponentially small splitting of the eigenvalues is to establish exponential decay estimates of Agmon type. These localization estimates are rather easy to obtain (at least to get the good scale in the exponential decay) in the corner cases due to the fact that the operator is "more elliptic" than in the regular case in the following sense: the spectral asymptotics is completely drifted by the principal symbol. Nevertheless, let us notice here that establishing the optimal estimates of Agmon is still an open problem. In smooth cases, due to a lack of ellipticity and to the multiple scales, the localization estimates obtained in the literature are in general not optimal at all (or rely on the presence of an electric potential, see [113, 114]): the principal symbol provides only a partial confinement whereas the precise localization of the eigenfunctions seems to be determined by the subprincipal terms. Our WKB analysis, in some explicit cases, give some hints for the optimal candidate to be the effective Agmon distance. The following result is proved in [BHR14].

Theorem 2.39. *We assume $A_2 = 0$ and A_1 is real analytic. Under Assumptions 2.32, 2.33 and 2.34, there exist a function $\Phi = \Phi(s)$ defined in a neighborhood $\mathcal{V}$ of x_0 with* $\mathsf{Re}\,\mathsf{Hess}\Phi(x_0) > 0$ *and, for any $n \geq 1$, a sequence of real numbers $(\lambda_{n,j})_{j\geq 0}$ such that*

$$\lambda_n(h) \underset{h\to 0}{\sim} \sum_{j\geq 0} \lambda_{n,j} h^j,$$

in the sense of formal series, with $\lambda_{n,0} = \mu_0$. Besides there exists a formal series of smooth functions on $\mathcal{V} \times \mathbb{R}^n_t$

$$a_n(.;h) \underset{h\to 0}{\sim} \sum_{j\geq 0} a_{n,j} h^j$$

with $a_{n,0} \neq 0$ such that

$$(\mathfrak{L}_h - \lambda_n(h)) \left(a_n(.;h) e^{-\Phi/h}\right) = \mathcal{O}\left(h^\infty\right) e^{-\Phi/h}.$$

Furthermore the functions $t \mapsto a_{n,j}(s,t)$ belong to the Schwartz class uniformly in $s \in \mathcal{V}$. In addition, if A_1 is a polynomial function, there exists $c_0 > 0$ such that for all $h \in (0, h_0)$

$$\mathcal{B}\Big(\lambda_{n,0} + \lambda_{n,1} h, c_0 h\Big) \cap \mathsf{sp}\left(\mathfrak{L}_h\right) = \{\lambda_n(h)\},$$

and $\lambda_n(h)$ is a simple eigenvalue.

In the previous theorem we used the following definition of formal series of functions.

Notation 2.40. *Let $n \geq 1$. We write $a_n(s,t;h) \underset{h\to 0}{\sim} \sum_{j\geq 0} a_{n,j}(s,t) h^j$ when for all $J \geq 0$ and $\alpha \in \mathbb{N}^{n+m}$, there exist $h_{J,\alpha} > 0$ and $C_{J,\alpha} > 0$ such that for all $h \in (0, h_{J,\alpha})$, we have*

$$\left| D^\alpha \Big(a_n(s,t;h) - \sum_{j=0}^{J} a_{n,j}(s,t) h^j \Big) \right| \leq C_{J,\alpha} h^{J+1} \quad \text{locally in } (s,t) \in \mathcal{V} \times \mathbb{R}^n.$$

We also write $a = \mathcal{O}(h^\infty)$ when $a \sim 0$. The case of formal series of numbers is similar.

Let us also recall that for any arbitrary sequence of smooth functions a_j one can always find, by a procedure of Borel type, a unique smooth function $a(s,t;h)$ (called a realization) (up to $\mathcal{O}(h^\infty)$) such that $a(s,t;h) \underset{h\to 0}{\sim} \sum_{j\geq 0} a_j(s,t)h^j$.

Remark 2.41. *When A_2 is not zero, it appears that the dimensional reduction is prevented by the oscillations of the eigenfunctions of the model operator $\mathcal{M}_{x,\xi}$. The problem already appears in the case $t\in\mathbb{R}$: we can gauge out A_2 at the price to replace A_1 by $A_1 + h\nabla_s\varphi(s,t)$ which is h dependent. As a consequence of our analysis, we can check that the spectrum associated with the potential $(A_1 + h\nabla_s\varphi, 0)$ is shifted by a factor $\mathcal{O}(h)$ compared to the one associated with $(A_1, 0)$. In dimension one for t, we can even prove that the phase Φ in the WKB expansion is (s,t)-dependent.*

Let us explain the main lines of the proof of Theorem 2.39. Thanks to Theorem 2.37, we have sharp asymptotic expansions of the eigenvalues. In particular, one knows that they become simple in the semiclassical limit. Therefore, to get the (WKB) approximation of the corresponding eigenfunctions, we have just to use an appropriate Ansatz for our quasimodes and to apply the spectral theorem. The new Ansatz considered here is given by a partial WKB expansion with respect to the variable s. Under our analyticity assumptions, the effective eikonal equation is solved thanks to the classical stable manifold theorem and analytic extensions of the eigenpairs of the "model" operators. The corresponding effective transport equation is obtained as the Fredholm condition of an operator valued transport equation jointly with the Feynman-Hellmann formulas.

3.3.2. *WKB expansions for $\mathfrak{L}_h^{\mathsf{vf},[k]}$.* The following theorem (which is almost an obvious consequence of Theorem 2.39) states that the first eigenfunctions of $\mathfrak{L}_h^{\mathsf{vf},[k]}$ (defined in (2.3.11)) are in the WKB form (and so are the eigenfunctions of the fully semiclassical magnetic Laplacian $\mathfrak{L}_{h,\mathbf{A}^{[k]}}$ which is the pilot operator in situations involving for instance an additional metric).

Theorem 2.42. *Let us assume that γ is analytic with $\liminf_{x\to\pm\infty}\gamma = \gamma_\infty \in (\gamma_0, +\infty]$. In the analytic case, there exist a function $\Phi = \Phi(s)$ defined in a neighborhood $\mathcal{V}$ of 0 with $\operatorname{Re}\Phi''(0) > 0$ and a sequence of real numbers $\lambda_{n,j}^{\mathsf{vf}}$ such that the n-th eigenvalue of $\mathfrak{L}_h^{\mathsf{vf},[k]}$ satisfies*

$$\lambda_n^{\mathsf{vf}}(h) \underset{h\to 0}{\sim} \sum_{j\geq 0} \lambda_{n,j}^{\mathsf{vf}} h^{\frac{j}{k+2}}$$

in the sense of formal series, with $\lambda_{n,0}^{\mathsf{vf}} = \mu_0 = \nu_1^{[k]}(\zeta_0^{[k]})$. Besides there exists a formal series of smooth functions on $\mathcal{V}\times\mathbb{R}_t^n$

$$a_n^{\mathsf{vf}}(.,h) \underset{h\to 0}{\sim} \sum_{j\geq 0} a_{n,j}^{\mathsf{vf}} h^{\frac{j}{k+2}}$$

with $a_{n,0}^{\mathsf{vf}} \neq 0$ such that

$$\left(\mathfrak{L}_h^{\mathsf{vf},[k]} - \lambda_n(h)\right)\left(a_n^{\mathsf{vf}}(\cdot,h)e^{-\Phi/h^{\frac{1}{k+2}}}\right) = \mathcal{O}\left(h^\infty\right)e^{-\Phi/h^{\frac{1}{k+2}}},$$

There exists $c_0 > 0$ such that for all $h \in (0, h_0)$

$$\mathcal{B}\left(\lambda^{\mathsf{vf}}_{n,0} + \lambda^{\mathsf{vf}}_{n,1} h^{\frac{1}{k+2}}, c_0 h^{\frac{2}{k+2}}\right) \cap \mathsf{sp}\left(\mathfrak{L}^{\mathsf{vf},[k]}_h\right) = \{\lambda^{\mathsf{vf}}_n(h)\},$$

and $\lambda^{\mathsf{vf}}_n(h)$ is a simple eigenvalue.

Remark 2.43. *If $\gamma(s)^{-1}\gamma(0) - 1$ is small enough (weak magnetic barrier), our construction of Φ can be made global, that is $\mathcal{V} = \mathbb{R}$. Note that the simplicity is a consequence of the analysis of* [**R13a**, **DoR13**] *which does not use that γ is a polynomial.*

3.3.3. *Along a varying edge.* Let us provide another example for which one can produce a WKB analysis but which is not a direct consequence of Theorem 2.39. This example, analyzed in [**BHR14**], is inspired by the collaboration with N. Popoff [**PR13**]. This one is motivated by the analysis of problems with singular boundaries. Here we are concerned with the case when the domain is a wedge with varying aperture, that is with the Neumann magnetic Laplacian $\mathfrak{L}^{\mathsf{e}}_{h,\mathbf{A}} = (-ih\nabla + \mathbf{A})^2$ on $\mathsf{L}^2(\mathcal{W}_{s\mapsto\alpha(s)}, \mathrm{d}s\,\mathrm{d}t\,\mathrm{d}z)$. Let us recall the definition of the magnetic wedge with constant aperture α. Many properties of this operator can be found in the thesis of Popoff [**121**]. We let

$$\mathcal{W}_\alpha = \mathbb{R} \times \mathcal{S}_\alpha,$$

where the 2D corner with fixed angle $\alpha \in (0, \pi)$ is defined by:

$$\mathcal{S}_\alpha = \left\{(t, z) \in \mathbb{R}^2 : |z| < t \tan\left(\frac{\alpha}{2}\right)\right\}.$$

Definition 2.44. *Let $\mathfrak{L}^{\mathsf{e}}_\alpha$ be the Neumann realization on $\mathsf{L}^2(\mathcal{W}_\alpha, \mathrm{d}s\,\mathrm{d}t\,\mathrm{d}z)$ of*

$$D_t^2 + D_z^2 + (D_s - t)^2. \tag{2.3.12}$$

We denote by $\nu^{\mathsf{e}}_1(\alpha)$ the bottom of the spectrum of $\mathfrak{L}^{\mathsf{e}}_\alpha$.

Using the Fourier transform with respect to $\hat{s}$, we have the decomposition:

$$\mathfrak{L}^{\mathsf{e}}_\alpha = \int^{\oplus} \mathfrak{L}^{\mathsf{e}}_{\alpha,\zeta}\,\mathrm{d}\zeta, \tag{2.3.13}$$

where $\mathfrak{L}^{\mathsf{e}}_{\alpha,\zeta}$ is the following Neumann realization on $\mathsf{L}^2(\mathcal{S}_\alpha, \mathrm{d}t\,\mathrm{d}z)$:

$$\mathfrak{L}^{\mathsf{e}}_{\alpha,\zeta} = D_t^2 + D_z^2 + (\zeta - t)^2, \tag{2.3.14}$$

where $\zeta \in \mathbb{R}$ is the Fourier parameter. As

$$\lim_{\substack{|(t,z)|\to+\infty \\ (t,z)\in\mathcal{S}_\alpha}} (\zeta - t)^2 = +\infty,$$

the Schrödinger operator $\mathfrak{L}^{\mathsf{e}}_{\alpha,\zeta}$ has compact resolvent for all $(\alpha, \zeta) \in (0, \pi) \times \mathbb{R}$.

Notation 2.45. *For each $\alpha \in (0, \pi)$, we denote by $\nu^{\mathsf{e}}_1(\alpha, \eta)$ the lowest eigenvalue of $\mathfrak{L}^{\mathsf{e}}_{\alpha,\zeta}$ and we denote by $u^{\mathsf{e}}_{\alpha,\zeta}$ a normalized corresponding eigenfunction.*

Using (2.3.13) we have:

$$\nu_1^{\mathsf{e}}(\alpha) = \inf_{\zeta \in \mathbb{R}} \nu_1^{\mathsf{e}}(\alpha, \zeta). \tag{2.3.15}$$

Let us gather a few elementary properties.

Lemma 2.46. *We have:*

(1) For all $(\alpha, \zeta) \in (0, \pi) \times \mathbb{R}$, $\nu_1^{\mathsf{e}}(\alpha, \zeta)$ *is a simple eigenvalue of* $\mathfrak{L}^{\mathsf{e}}_{\alpha,\zeta}$.
(2) The function $(0, \pi) \times \mathbb{R} \ni (\alpha, \zeta) \mapsto \nu_1^{\mathsf{e}}(\alpha, \zeta)$ *is analytic.*
(3) For all $\zeta \in \mathbb{R}$, *the function* $(0, \pi) \ni \alpha \mapsto \nu_1^{\mathsf{e}}(\alpha, \zeta)$ *is decreasing.*
(4) The function $(0, \pi) \ni \alpha \mapsto \nu_1^{\mathsf{e}}(\alpha)$ *is non increasing.*
(5) For all $\alpha \in (0, \pi)$, *we have*

$$\lim_{\eta \to -\infty} \nu_1^{\mathsf{e}}(\alpha, \zeta) = +\infty \quad \textit{and} \quad \lim_{\zeta \to +\infty} \nu_1^{\mathsf{e}}(\alpha, \zeta) = \mathfrak{s}(\tfrac{\pi - \alpha}{2}). \tag{2.3.16}$$

Proof. We refer to [121, Section 3] for the first two statements. The monotonicity comes from [121, Proposition 8.14] and the limits as ζ goes to $\pm\infty$ are computed in [121, Theorem 5.2]. □

Remark 2.47. *As* $\nu_1^{\mathsf{e}}(\pi) = \Theta_0$, *we have:*

$$\forall \alpha \in (0, \pi), \quad \nu_1^{\mathsf{e}}(\alpha) \geq \Theta_0. \tag{2.3.17}$$

Let us note that it is proved in [121, Proposition 8.13] *that* $\nu_1^{\mathsf{e}}(\alpha) > \Theta_0$ *for all* $\alpha \in (0, \pi)$.

Proposition 2.48. *There exists* $\tilde{\alpha} \in (0, \pi)$ *such that for* $\alpha \in (0, \tilde{\alpha})$, *the function* $\zeta \mapsto \nu_1^{\mathsf{e}}(\alpha, \zeta)$ *reaches its infimum and*

$$\nu_1^{\mathsf{e}}(\alpha) < \mathfrak{s}\left(\frac{\pi - \alpha}{2}\right), \tag{2.3.18}$$

where the spectral function $\mathfrak{s}$ *is defined in Chapter 1, Section 1.4.4.*

Remark 2.49. *By computing* C^{qm}, *we notice that* (2.3.18) *holds at least for* $\alpha \in (0, 1.2035)$. *Numerical computations show that in fact* (2.3.18) *seems to hold for all* $\alpha \in (0, \pi)$.

We will work under the following conjecture:

Conjecture 2.50. *For all* $\alpha \in (0, \pi)$, $\zeta \mapsto \nu_1^{\mathsf{e}}(\alpha, \zeta)$ *has a unique critical point denoted by* $\zeta_0^{\mathsf{e}}(\alpha)$ *and it is a non degenerate minimum.*

Remark 2.51. *A numerical analysis seems to indicate that Conjecture 2.50 is true (see* [121, Subsection 6.4.1]*).*

Under this conjecture and using the analytic implicit functions theorem, we deduce (see [PR13]):

Lemma 2.52. *Under Conjecture 2.50, the function* $(0, \pi) \ni \alpha \mapsto \zeta_0^{\mathsf{e}}(\alpha)$ *is analytic and so is* $(0, \pi) \ni \alpha \mapsto \nu_1^{\mathsf{e}}(\alpha)$. *Moreover the function* $(0, \pi) \ni \alpha \mapsto \nu_1^{\mathsf{e}}(\alpha)$ *is decreasing.*

We will assume that there is a unique point of maximal aperture (which is non-degenerate).

Assumption 2.53. *The function $s \mapsto \alpha(s)$ admits a unique and non-degenerate maximum α_0 at $s = 0$.*

Notation 2.54. *We let $\mathcal{T}(s) = \tan\left(\frac{\alpha(s)}{2}\right)$.*

In order to perform the WKB analysis in the wedge case, we need to consider the Neumann realization of the operator defined on $\mathsf{L}^2(\mathcal{S}_{\alpha_0}, \mathrm{d}t\,\mathrm{d}z)$ by

$$\mathcal{M}^{\mathsf{e}}_{s,\zeta} = D_t^2 + \mathcal{T}(s)^{-2}\mathcal{T}(0)^2 D_z^2 + (\zeta - t)^2,$$

whose form domain is

$$\mathsf{Dom}\,(\mathcal{Q}^{\mathsf{e}}_{s,\zeta}) = \left\{\psi \in \mathsf{L}^2(\mathcal{S}_{\alpha_0}) : D_t\psi \in \mathsf{L}^2(\mathcal{S}_{\alpha_0}), D_z\psi \in \mathsf{L}^2(\mathcal{S}_{\alpha_0}), t\psi \in \mathsf{L}^2(\mathcal{S}_{\alpha_0})\right\}$$

and with operator domain

$$\mathsf{Dom}\,(\mathcal{M}^{\mathsf{e}}_{s,\zeta}) = \left\{\psi \in \mathsf{Dom}\,(\mathcal{Q}^{\mathsf{e}}_{s,\zeta}) : \mathcal{M}^{\mathsf{e}}_{s,\zeta}\psi \in \mathsf{L}^2(\mathcal{S}_{\alpha_0}), \mathfrak{C}(s)\psi = 0\right\},$$

where

$$\mathfrak{C}(s) = -\mathsf{sgn}(z)D_t + \mathcal{T}(s)^{-2}\mathcal{T}(0)D_z.$$

The lowest eigenvalue of $\mathcal{M}^{\mathsf{e}}_{s,\zeta}$ is denoted by $\mu^{\mathsf{e}}(s,\zeta)$ and the corresponding normalized eigenfunction $u^{\mathsf{e}}_{s,\zeta}$. Conjecture 2.50 can be reformulated as follows.

Conjecture 2.55. *For all $\alpha_0 \in (0,\pi)$, the function $\zeta \mapsto \mu^{\mathsf{e}}(0,\zeta)$ admits a unique critical point ζ^{e}_0 which is a non-degenerate minimum.*

The following proposition essentially shows that the operator symbol $\mathcal{M}^{\mathsf{e}}_{s,\zeta}$ satisfies generic properties as in Assumption 2.32.

Proposition 2.56. *Under Assumption 2.53 and if Conjecture 2.55 is true, the function μ^{e} admits a local non-degenerate minimum at $(0,\zeta^{\mathsf{e}}_0)$. Moreover the Hessian at $(0,\zeta^{\mathsf{e}}_0)$ is given by*

$$4\kappa\mathcal{T}(0)^{-1}\|D_z u^{\mathsf{e}}_{0,\zeta^{\mathsf{e}}_0}\|s^2 + \partial^2_\zeta\mu^{\mathsf{e}}(0,\zeta^{\mathsf{e}}_0)\zeta^2, \tag{2.3.19}$$

where $\kappa = -\frac{\mathcal{T}''(0)}{2}$.

Thanks to this proposition, in [**BHR14**] we provide local (near the point of the edge giving the maximal aperture) WKB expansions of the lowest eigenfunctions.

3.3.4. *Curvature induced magnetic bound states.* As we have seen, in many situations the spectral splitting appears in the second term of the asymptotic expansion of the eigenvalues. It turns out that we can also deal with more degenerate situations. The next lines are motivated by the initial paper [**75**] whose main result is recalled in (1.1.2). This fundamental result establishes that a smooth Neumann boundary can trap the lowest eigenfunctions near the points of maximal curvature. These considerations are generalized in [51, Theorem 1.1] where the complete asymptotic expansion of the n-th eigenvalue of $\mathfrak{L}^{\mathsf{c}}_{h,\mathbf{A}} = (-ih\nabla + \mathbf{A})^2$ is provided and satisfies in particular:

$$\Theta_0 h - C_1\kappa_{\max}h^{3/2} + (2n-1)C_1\Theta_0^{1/4}\sqrt{\frac{3k_2}{2}}h^{7/4} + o(h^{7/4}), \tag{2.3.20}$$

where $k_2 = -\kappa''(0)$. As in [51], we consider the magnetic Neumann Laplacian on a smooth domain Ω such that the algebraic curvature κ satisfies the following assumption.

Assumption 2.57. *The function κ is smooth and admits a unique and non-degenerate maximum.*

In [BHR14] we prove that the lowest eigenfunctions are approximated by local WKB expansions which can be made global when for instance $\partial\Omega$ is the graph of a smooth function. In particular we recover the term $C_1\Theta_0^{1/4}\sqrt{\frac{3k_2}{2}}$ by a method different from the one of Fournais and Helffer and we explicitly provide a candidate to be the optimal distance of Agmon in the boundary. Since it is quite unusual to exhibit a pure magnetic Agmon distance, let us provide a precise statement. For that purpose, let us consider the following Neumann realization on $\mathsf{L}^2(\mathbb{R}^2_+, m(s,t)\,\mathrm{d}s\,\mathrm{d}t)$, which is nothing but the expression of the magnetic Laplacian in curvilinear coordinates,

$$\begin{aligned}\mathcal{L}^{\mathsf{c}}_h &= m(s,t)^{-1}hD_t m(s,t)hD_t \\ &+ m(s,t)^{-1}\left(hD_s + \zeta_0 h^{\frac12} - t + \kappa(s)\frac{t^2}{2}\right)m(s,t)^{-1}\left(hD_s + \zeta_0 h^{\frac12} - t + \kappa(s)\frac{t^2}{2}\right),\end{aligned} \tag{2.3.21}$$

where $m(s,t) = 1 - t\kappa(s)$. Thanks to the rescaling

$$t = h^{1/2}\tau, \qquad s = \sigma,$$

and after division by h the operator $\mathcal{L}^{\mathsf{c}}_h$ becomes

$$\begin{aligned}\mathfrak{L}^{\mathsf{c}}_h &= m(\sigma, h^{1/2}\tau)^{-1}D_\tau m(\sigma,h^{1/2}\tau)D_\tau \\ &+ m(\sigma,h^{1/2}\tau)^{-1}\left(h^{1/2}D_\sigma + \zeta_0 - \tau + h^{1/2}\kappa(\sigma)\frac{\tau^2}{2}\right)m(\sigma,h^{1/2}\tau)^{-1}\left(h^{1/2}D_\sigma + \zeta_0 - \tau + h^{1/2}\kappa(\sigma)\frac{\tau^2}{2}\right),\end{aligned} \tag{2.3.22}$$

on the space $\mathsf{L}^2(m(\sigma,h^{1/2}\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau)$.

Theorem 2.58. *Under Assumption 2.53, there exist a function*

$$\Phi = \Phi(\sigma) = \left(\frac{2C_1}{\nu_1''(\zeta_0)}\right)^{1/2}\left|\int_0^\sigma (\kappa(0)-\kappa(s))^{1/2}\,\mathrm{d}s\right|$$

defined in a neighborhood $\mathcal{V}$ of $(0,0)$ such that $\operatorname{Re}\Phi''(0) > 0$, and a sequence of real numbers $(\lambda^{\mathsf{c}}_{n,j})_{j\geq 0}$ such that

$$\lambda^{\mathsf{c}}_n(h) \underset{h\to 0}{\sim} \sum_{j\geq 0}\lambda^{\mathsf{c}}_{n,j}h^{\frac{j}{4}}.$$

Besides there exists a formal series of smooth functions on $\mathcal{V}$,

$$a_n^{\mathsf{c}} \underset{h\to 0}{\sim} \sum_{j\geq 0} a_{n,j}^{\mathsf{c}} h^{\frac{j}{4}}$$

such that

$$\left(\mathfrak{L}_h^{\mathsf{c}} - \lambda_n^{\mathsf{c}}(h)\right)\left(a_n^{\mathsf{c}} e^{-\Phi/h^{\frac{1}{4}}}\right) = \mathcal{O}\left(h^{\infty}\right) e^{-\Phi/h^{\frac{1}{4}}}.$$

We also have that $\lambda_{n,0}^{\mathsf{c}} = \Theta_0$, $\lambda_{n,1}^{\mathsf{c}} = 0$, $\lambda_{n,1}^{\mathsf{c}} = -C_1\kappa_{\max}$ and $\lambda_{n,3}^{\mathsf{c}} = (2n-1)C_1\Theta_0^{1/4}\sqrt{\frac{3k_2}{2}}$. The main term in the Ansatz is in the form

$$a_{n,0}^{\mathsf{c}}(\sigma,\tau) = f_{n,0}^{\mathsf{c}}(\sigma)u_{\zeta_0}(\tau).$$

Moreover, for all $n \geq 1$, there exist $h_0 > 0$, $c > 0$ such that for all $h \in (0, h_0)$, we have

$$\mathcal{B}\Big(\lambda_{n,0}^{\mathsf{c}} + \lambda_{n,2}^{\mathsf{c}} h^{1/2} + \lambda_{n,3}^{\mathsf{c}} h^{\frac{3}{4}}, ch^{\frac{3}{4}}\Big) \cap \mathsf{sp}\left(\mathfrak{L}_h^{\mathsf{c}}\right) = \{\lambda_n^{\mathsf{c}}(h)\},$$

and $\lambda_n^{\mathsf{c}}(h)$ is a simple eigenvalue.

Remark 2.59. *In particular, Theorem 2.58 proves that there are no odd powers of $h^{\frac{1}{8}}$ in the expansion of the eigenvalues (compare with [**51**, Theorem 1.1]).*

3.3.5. *Some numerical simulations.* We provide below some numerical simulations from [BHR14] (κ has one or two maxima).

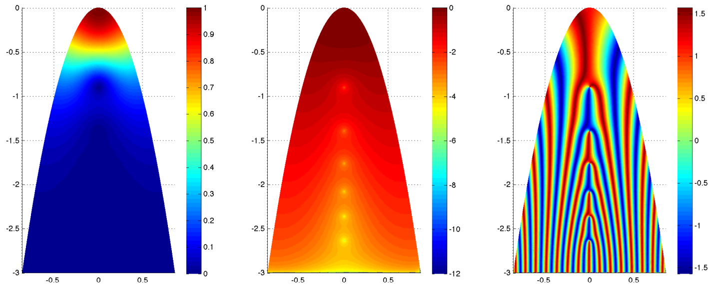
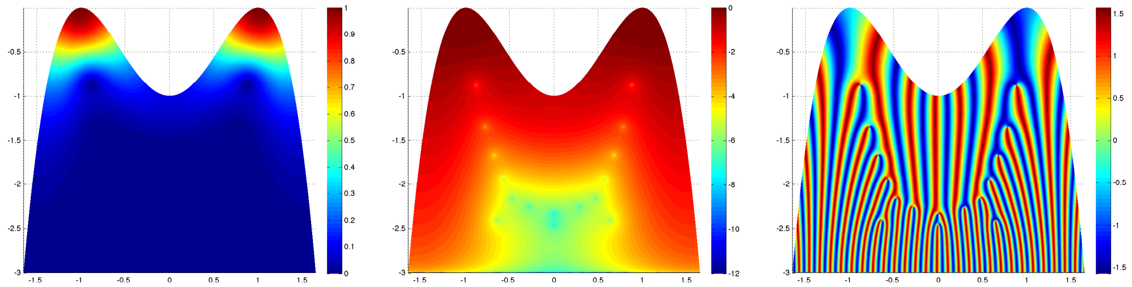
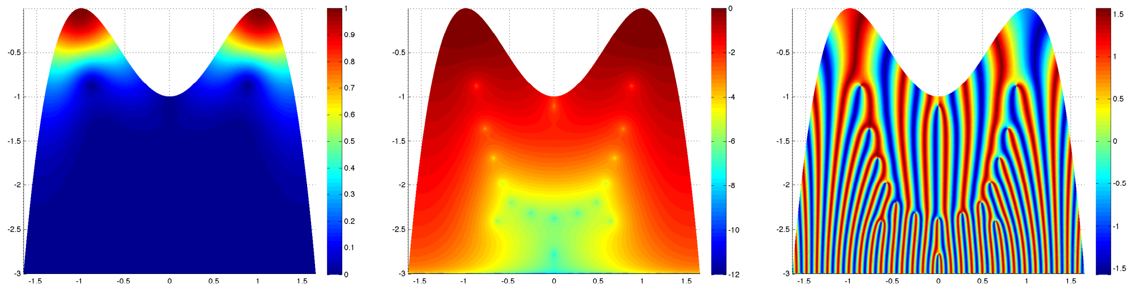


FIGURE 4. Modulus, $\log_{10}$(modulus) and phase of the first eigenfunction, $h = \frac{1}{20}$.



(a) First eigenvector



(b) Second eigenvector

FIGURE 5. Moduli, $\log_{10}$(moduli) and phases of the first two eigenvectors, $h = \frac{1}{20}$.

CHAPTER 3

Semiclassical magnetic normal forms

> Now do you imagine he would have attempted to inquire or learn what he thought he knew, when he did not know it, until he had been reduced to the perplexity of realizing that he did not know, and had felt a craving to know?
>
> *Meno*, Plato

In this chapter we highlight the normal form philosophy explained in Chapter 1, Section 3 by presenting four results of *magnetic harmonic approximation*. As we will see, each situation will present its specific features and difficulties:

- How can we deal with a vanishing magnetic field in dimension two? ([**DoR13**])
- How can we treat a problem with smooth boundary in dimension three? ([**R12**])
- Can we still display a precise semiclassical asymptotics in dimension three if the boundary is not smooth? ([**PR13**])
- In dimension two and without boundary, can we describe more than $\lambda_n(h)$ for fixed n? ([**RVN14**])

1. Vanishing magnetic fields in dimension two

In this section we study the influence of the cancellation of the magnetic field along a smooth curve in dimension two. The results of this section are joint work with N. Dombrowski [**DoR13**].

1.1. Framework. We consider a vector potential $\mathbf{A} \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R}^2)$ and we consider the self-adjoint operator on $\mathsf{L}^2(\mathbb{R}^2)$ defined by:

$$\mathfrak{L}_{h,\mathbf{A}} = (-ih\nabla + \mathbf{A})^2.$$

1.1.1. *How does* $\boldsymbol{B}$ *vanish?* In order $\mathfrak{L}_{h,\mathbf{A}}$ to have compact resolvent, we will assume that:

$$\mathbf{B}(x) \underset{|x|\to+\infty}{\to} +\infty. \tag{3.1.1}$$

Notation 3.1. *We will denote by* $\lambda_n(h)$ *the* n*-th eigenvalue of* $\mathfrak{L}_{h,\boldsymbol{A}}$.

As in [119, 68], we will investigate the case when $\mathbf{B}$ cancels along a closed and smooth curve $\mathcal{C}$ in $\mathbb{R}^2$. We have already discussed the motivation in Chapter 2, Section 2. Let us notice that the assumption (3.1.1) could clearly be relaxed so that one could also consider a smooth, bounded and simply connected domain of $\mathbb{R}^2$ with Dirichlet or Neumann condition on the boundary as far as the magnetic field does not vanish near the boundary (in this case one should meet a model presented in Chapter 2, Section 2). We let:

$$\mathcal{C} = \{c(s), s \in \mathbb{R}\}.$$

We assume that $\mathbf{B}$ is positive inside $\mathcal{C}$ and negative outside. We introduce the standard tubular coordinates (s, t) near $\mathcal{C}$ defined by the map

$$(s, t) \mapsto c(s) + t\mathbf{n}(s),$$

where $\mathbf{n}(s)$ denotes the inward pointing normal to $\mathcal{C}$ at $c(s)$. The function $\tilde{\mathbf{B}}$ will denote $\mathbf{B}$ in the coordinates (s, t), so that $\tilde{\mathbf{B}}(s, 0) = 0$.

1.1.2. *Heuristics and leading operator.* Let us adopt first a heuristic point of view to introduce the leading operator of the analysis presented in this section. We want to describe the operator $\mathfrak{L}_{h,\mathbf{A}}$ near the cancellation line of $\mathbf{B}$, that is near $\mathcal{C}$. In a rough approximation, near $(s_0, 0)$, we can imagine that the line is straight $(t = 0)$ and that the magnetic field cancels linearly so that we can consider $\tilde{\mathbf{B}}(s, t) = \gamma(s_0)t$ where $\gamma(s_0)$ is the derivative of $\tilde{\mathbf{B}}$ with respect to t. Therefore the operator to which we are reduced at the leading order near s_0 is:

$$h^2 D_t^2 + \left(hD_s - \gamma(s_0)\frac{t^2}{2}\right)^2.$$

This operator is a special case of the larger class introduced in Chapter 2.

1.2. Montgomery operator and rescaling. We will be led to use the Montgomery operator with parameters $\eta \in \mathbb{R}$ and $\gamma > 0$:

$$\mathfrak{L}^{[1]}_{\gamma,\zeta} = D_t^2 + \left(\zeta - \frac{\gamma}{2}t^2\right)^2. \tag{3.1.2}$$

The Montgomery operator has clearly compact resolvent and we can consider its lowest eigenvalue denoted by $\nu_1^{[1]}(\gamma, \zeta)$. In fact one can take $\gamma = 1$ up to the rescaling $t = \gamma^{-1/3}\tau$ and $\mathfrak{L}^{[1]}_{\gamma,\zeta}$ is unitarily equivalent to:

$$\gamma^{2/3}\left(D_\tau^2 + (-\eta\gamma^{-1/3} + \frac{1}{2}\tau^2)^2\right) = \gamma^{2/3}\mathfrak{L}^{[1]}_{1,\zeta\gamma^{-1/3}}.$$

Let us emphasize that this rescaling is related to the normal form analysis that we use in the semiclassical spectral asymptotics. For all $\gamma > 0$, we have (see Chapter 2, Proposition 2.5):

(3.1.3) $\zeta \mapsto \nu_1^{[1]}(\gamma, \zeta)$ admits a unique and non-degenerate minimum at a point $\zeta_0^{[1]}(\gamma)$.

If $\gamma = 1$, we have $\zeta_0^{[1]}(1) = \zeta_0^{[1]}$. We may write:

$$\inf_{\zeta\in\mathbb{R}} \nu_1^{[1]}(\gamma,\zeta) = \gamma^{2/3}\nu_1^{[1]}(\zeta_0^{[1]}). \tag{3.1.4}$$

Let us recall some notation.

Notation 3.2. *We notice that $\mathfrak{L}_\zeta^{[1]} = \mathfrak{L}_{1,\zeta}^{[1]}$ and we denote by $u_\zeta^{[1]}$ a L^2-normalized and positive eigenfunction associated with $\nu_1^{[1]}(\zeta)$.*

For fixed $\gamma > 0$, the family $(\mathfrak{L}_{\gamma,\zeta}^{[1]})_{\eta\in\mathbb{R}}$ is an analytic family of type (B) so that the eigenpair $(\nu_1^{[1]}(\zeta), u_\zeta^{[1]})$ has an analytic dependence on ζ (see [89]).

1.3. Semiclassical asymptotics with vanishing magnetic fields. We consider the normal derivative of $\mathbf{B}$ on $\mathcal{C}$, i.e. the smooth function $\gamma : s \mapsto \partial_t\tilde{\mathbf{B}}(s,0)$. We will assume that:

Assumption 3.3. *γ admits a unique, non-degenerate and positive minimum at $s_0 = 0$.*

We let $\gamma_0 = \gamma(0)$. Let us state the main result of this section:

Theorem 3.4. *We assume Assumption 3.3. For all $n \geq 1$, there exists a sequence $(\theta_j^n)_{j\geq 0}$ such that we have:*

$$\lambda_n(h) \underset{h\to 0}{\sim} h^{4/3}\sum_{j\geq 0}\theta_j^n h^{j/6}$$

where:

$$\theta_0^n = \gamma_0^{2/3}\nu_1^{[1]}(\zeta_0^{[1]}), \quad \theta_1^n = 0, \quad \theta_2^n = \gamma_0^{2/3}C_0 + \gamma_0^{2/3}(2n-1)\left(\frac{\alpha\nu_1^{[1]}(\zeta_0^{[1]})(\nu_1^{[1]})''(\zeta_0^{[1]})}{3}\right)^{1/2},$$

where we have let:

$$\alpha = \frac{1}{2}\gamma_0^{-1}\gamma''(0) > 0 \tag{3.1.5}$$

and:

$$C_0 = \langle Lu_{\zeta_0^{[1]}}^{[1]}, u_{\zeta_0^{[1]}}^{[1]}\rangle_{\mathsf{L}^2(\mathbb{R}_{\hat{\tau}})}, \tag{3.1.6}$$

where:

$$L = 2k(0)\gamma_0^{-4/3}\left(\frac{\hat{\tau}^2}{2} - \zeta_0^{[1]}\right)\hat{\tau}^3 + 2\hat{\tau}\gamma_0^{-1/3}\kappa(0)\left(-\zeta_0^{[1]} + \frac{\hat{\tau}^2}{2}\right)^2,$$

and:

$$k(0) = \frac{1}{6}\partial_t^2\tilde{\boldsymbol{B}}(0,0) - \frac{\kappa(0)}{3}\gamma_0.$$

Remark 3.5. *This theorem is mainly motivated by the paper of Helffer and Kordyukov [68] (see also [66,* Section 5.2] *where the above result is presented as a conjecture and the paper [74] where the case of discrete wells is analyzed) where the authors prove a one term asymptotics for all the eigenvalues (see [68,* Corollary 1.1]*). Moreover, they also prove an accurate upper bound in [68,* Theorem 1.4] *thanks to a Grushin type method* *(see [64]). This result could be generalized to the case when the magnetic vanishes on hypersurfaces at a given order. By using the results of [BHR14], we can improve the construction of quasimodes done in [DoR13] into a WKB construction.*

2. Variable magnetic field and smooth boundary in dimension three

This section is devoted to the investigation of the relation between the boundary and the magnetic field in dimension three. We will see that the semiclassical structure is completely different from the one presented in the previous section even if the eigenvalues expansions look the same. The different results are obtained in [R12].

2.1. A toy operator with variable magnetic field. Let us introduce the geometric domain

$$\Omega_0 = \{(x,y,z) \in \mathbb{R}^3 : |x| \leq x_0, |y| \leq y_0 \text{ and } 0 < z \leq z_0\},$$

where $x_0, y_0, z_0 > 0$. The part of the boundary which carries the Dirichlet condition is given by

$$\partial_{\mathsf{Dir}}\Omega_0 = \{(x,y,z) \in \Omega_0 : |x| = x_0 \text{ or } |y| = y_0 \text{ or } z = z_0\}.$$

2.1.1. *Definition of the operator.* For $h > 0$, $\alpha \geq 0$ and $\theta \in \left(0, \frac{\pi}{2}\right)$, we consider the self-adjoint operator:

$$\mathfrak{L}_{h,\alpha,\theta} = h^2 D_y^2 + h^2 D_z^2 + (hD_x + z\cos\theta - y\sin\theta + \alpha z(x^2+y^2))^2, \tag{3.2.1}$$

with domain:

$$\begin{aligned}\mathsf{Dom}\,(\mathfrak{L}_{h,\alpha,\theta}) = \{\psi \in L^2(\Omega_0) : \quad & \mathfrak{L}_{h,\alpha,\theta}\psi \in L^2(\Omega_0),\\ & \psi = 0 \quad \text{on} \quad \partial_{\mathsf{Dir}}\Omega_0 \quad \text{and} \quad \partial_z\psi = 0 \text{ on } z = 0\}.\end{aligned}$$

We denote by $(\lambda(h), u_h)$ an eigenpair and we let $\mathfrak{L}_h = \mathfrak{L}_{h,\alpha,\theta}$ (we omit the dependence on α and θ). The vector potential is expressed as:

$$\mathbf{A}(x,y,z) = (V_\theta(y,z) + \alpha z(x^2+y^2), 0, 0)$$

where

$$V_\theta(y,z) = z\cos\theta - y\sin\theta. \tag{3.2.2}$$

The associated magnetic field is given by:

$$\nabla \times \mathbf{A} = \mathbf{B} = (0, \cos\theta + \alpha(x^2+y^2), \sin\theta - 2\alpha yz). \tag{3.2.3}$$

2.1.2. *Constant magnetic field ($\alpha = 0$).* Let us examine the important case when $\alpha = 0$:

$$\mathfrak{L}_{h,0,\theta} = h^2 D_y^2 + h^2 D_z^2 + (hD_x + V_\theta(y,z))^2,$$

viewed as an operator on $\mathsf{L}^2(\mathbb{R}^3_+)$. We perform the rescaling:

$$x = h^{1/2}r, \quad y = h^{1/2}s, \quad z = h^{1/2}t \tag{3.2.4}$$

and the operator becomes (after division by h):

$$\mathfrak{L}_{1,0,\theta} = D_s^2 + D_t^2 + (D_r + V_\theta(s,t))^2.$$

Making a Fourier transform in the variable r denoted by $\mathcal{F}$, we get:

$$\mathcal{F}\mathfrak{L}_{1,0,\theta}\mathcal{F}^{-1} = D_s^2 + D_t^2 + (\eta + V_\theta(s,t))^2. \tag{3.2.5}$$

Then, we use a change of coordinates:

$$U_\theta(\eta, s, t) = (\rho, \sigma, \tau) = \left(\eta, s - \frac{\eta}{\sin\theta}, t\right) \tag{3.2.6}$$

and we obtain:

$$\mathfrak{H}_\theta^{\mathsf{Neu}} = U_\theta\mathcal{F}\mathfrak{L}_{1,0,\theta}\mathcal{F}^{-1}U_\theta^{-1} = D_\sigma^2 + D_\tau^2 + V_\theta(\sigma,\tau)^2.$$

Notation 3.6. *We denote by $\mathfrak{Q}_\theta^{\mathsf{Neu}}$ the quadratic form associated with $\mathfrak{H}_\theta^{\mathsf{Neu}}$.*

The operator $\mathfrak{H}_\theta^{\mathsf{Neu}}$ viewed as an operator acting on $\mathsf{L}^2(\mathbb{R}^2_+)$ is nothing but $\mathfrak{L}_\theta^{\mathsf{LP}}$ (see Chapter 1, Section 1.4.4). Let us also recall that the lower bound of the essential spectrum is related, through the Persson's theorem, to the following estimate:

$$\mathfrak{q}_\theta^{\mathsf{LP}}(\chi_R u) \geq (1 - \varepsilon(R))\|\chi_R u\|, \quad \forall u \in \mathsf{Dom}\,(\mathfrak{q}_\theta^{\mathsf{LP}}),$$

where $\mathfrak{q}_\theta^{\mathsf{LP}}$ is the quadratic form associated with $\mathfrak{L}_\theta^{\mathsf{LP}}$, where χ_R is a cutoff function away from the ball $B(0,R)$ and $\varepsilon(R)$ is tending to zero when R tends to infinity. Moreover, if we consider the Dirichlet realization $\mathfrak{L}_\theta^{\mathsf{LP,Dir}}$, we have:

$$\mathfrak{q}_\theta^{\mathsf{LP,Dir}}(u) \geq \|u\|^2, \quad \forall u \in \mathsf{Dom}\,(\mathfrak{q}_\theta^{\mathsf{LP,Dir}}). \tag{3.2.7}$$

2.1.3. *A "generic" model.* Let us explain why we are led to consider our model. Let us introduce a fundamental invariant in the case of variable magnetic field and our generic assumptions. We let:

$$\hat{\mathbf{B}}(x,y) = \mathfrak{s}(\theta(x,y))\|\mathbf{B}(x,y,0)\|,$$

where $\theta(x,y)$ is the angle of $\mathbf{B}(x,y,0)$ with the boundary $z=0$:

$$\|\mathbf{B}(x,y,0)\|\sin\theta(x,y) = \mathbf{B}(x,y,0)\cdot\mathbf{n}(x,y),$$

where $\mathbf{n}(x,y)$ is the inward normal at $(x,y,0)$. It is proved in [101] that the semiclassical asymptotics of the lowest eigenvalue is:

$$\lambda_1(h) = \min(\inf_{z=0}\hat{\mathbf{B}}, \inf_{\Omega_0}\|\mathbf{B}\|)h + o(h).$$

We are interested in the case when the following generic assumptions are satisfied:

$$\inf_{z=0}\hat{\mathbf{B}} < \inf_{\Omega_0}\|\mathbf{B}\| \tag{3.2.8}$$

$$\hat{\mathbf{B}} \text{ admits a unique and non degenerate minimum.} \tag{3.2.9}$$

Under these assumptions, a three terms upper bound is proved for $\lambda_1(h)$ in [**R10c**] and the corresponding lower bound, for a general domain, is still an open problem.

For $\alpha > 0$, the toy operator (3.2.1) is the simplest example of a generic Schrödinger operator with variable magnetic field satisfying Assumptions (3.2.8) and (3.2.9). We have the Taylor expansion:

$$\hat{\mathbf{B}}(x,y) = \mathfrak{s}(\theta) + \alpha C(\theta)(x^2+y^2) + O(|x|^3+|y|^3). \tag{3.2.10}$$

with:

$$C(\theta) = \cos\theta\mathfrak{s}(\theta) - \sin\theta\mathfrak{s}'(\theta).$$

Moreover, it is proved in [BDPR12] that $C(\theta) > 0$, for $\theta \in \left(0, \frac{\pi}{2}\right)$. Thus, Assumption (3.2.9) is verified if x_0, y_0 and z_0 are fixed small enough. Using $\mathfrak{s}(\theta) < 1$ when $\theta \in \left(0, \frac{\pi}{2}\right)$ and $\|\mathbf{B}(0,0,0)\| = 1$, we get Assumption (3.2.8).

2.1.4. *Remark on the function $\hat{\boldsymbol{B}}$.* Using the explicit expression of the magnetic field, we have:

$$\hat{\mathbf{B}}(x,y) = \hat{\mathbf{B}}_{\mathsf{rad}}(R), \quad R = \alpha(x^2+y^2)$$

and an easy computation gives:

$$\hat{\mathbf{B}}_{\mathsf{rad}}(R) = \|\mathbf{B}_{\mathsf{rad}}(R)\|\mathfrak{s}\left(\arctan\left(\frac{\sin\theta}{\cos\theta+R}\right)\right),$$

with

$$\|\mathbf{B}_{\mathsf{rad}}(R)\| = \sqrt{(\cos\theta+R)^2+\sin^2\theta}.$$

The results of [BDPR12] imply that $\hat{\mathbf{B}}_{\mathsf{rad}}$ is strictly increasing and

$$\partial_R\hat{\mathbf{B}}_{\mathsf{rad}}(R=0) = C(\theta) > 0.$$

Consequently, $\hat{\mathbf{B}}$ admits a unique and non degenerate minimum on $\mathbb{R}^3_+$ and tends to infinity far from 0. This is easy to see that:

$$\inf_{\mathbb{R}^3_+} \|\mathbf{B}\| = \cos\theta.$$

We deduce that, as long as $\mathfrak{s}(\theta) < \cos\theta$, the generic assumptions are satisfied with $\Omega_0 = \mathbb{R}^3_+$.

2.2. Three dimensional magnetic wells induced by the magnetic field and the (smooth) boundary. Let us introduce the fundamental operator

$$\mathfrak{S}_\theta(D_\rho,\rho) = \left(2\int_{\mathbb{R}^2_+} \tau V_\theta(u_\theta^{\mathsf{LP}})^2 \,\mathrm{d}\sigma\,\mathrm{d}\tau\right)\mathcal{H}_{\mathsf{harm}} + \left(\frac{2}{\sin\theta}\int_{\mathbb{R}^2_+}\tau V_\theta(u_\theta^{\mathsf{LP}})^2\,\mathrm{d}\sigma\,\mathrm{d}\tau\right)\rho + d(\theta),$$

where

$$\mathcal{H}_{\mathsf{harm}} = D_\rho^2 + \frac{\rho^2}{\sin^2\theta}$$

and

$$d(\theta) = \sin^{-2}\theta\langle\tau(D_\sigma^2V_\theta + V_\theta D_\sigma^2)u_\theta^{\mathsf{LP}}, u_\theta^{\mathsf{LP}}\rangle + 2\int_{\mathbb{R}^2_+}\tau\sigma^2V_\theta(u_\theta^{\mathsf{LP}})^2\,\mathrm{d}\sigma\,\mathrm{d}\tau.$$

We recall the important fact that (see [**R10c**, Formula (2.31)]):

$$2\int_{\mathbb{R}^2_+} tV_\theta(u_\theta^{\mathsf{LP}})^2\,\mathrm{d}s\,\mathrm{d}t = C(\theta) > 0,$$

so that $\mathfrak{S}_\theta(D_\rho,\rho)$ can be viewed as the harmonic oscillator up to dilation and translations.

We can now state the main result of this section.

Theorem 3.7. *For all $\alpha > 0$, $\theta \in \left(0, \frac{\pi}{2}\right)$, there exist a sequence $(\mu_{j,n})_{j\geq 0}$ and $\varepsilon_0 > 0$ s. t. for $|x_0| + |y_0| + |z_0| \leq \varepsilon_0$,*

$$\lambda_n(h) \sim h\sum_{j\geq 0}\mu_{j,n}h^j$$

and we have $\mu_{0,n} = \mathfrak{s}(\theta)$ and $\mu_{1,n}$ is the n-th eigenvalue of $\mathfrak{S}_\theta(D_\rho,\rho)$.

Remark 3.8. *The proof of Theorem 3.7 relies on the proof of accurate microlocal properties of the eigenfunctions and especially we use, in [**R12**], multiple commutator estimates to get precise polynomial bound of the eigenfunctions in the phase space. Somehow this strategy can remind the spirit of hypoellipticity.*

3. When a magnetic field meets a curved edge

We analyze here the effect of an edge in the boundary and how its combines with the magnetic field to produce a spectral asymptotics. This was the aim of the collaboration with N. Popoff [**PR13**].

3.1. Geometrical assumptions and local models.

3.1.1. *Description of the lens.* We first define the lens Ω.

Definition 3.9. *Let Σ be a smooth and connected surface in $\mathbb{R}^3$ and Π be the plane $x_3 = 0$. We assume that the intersection $\Sigma \cap \Pi$ is a smooth and closed curve and that Σ and Π intersect neither normally nor tangentially. Denoting by Σ^+ the set $\{\mathbf{x} \in \Sigma : x_3 > 0\}$ and by Σ^- its symmetric with respect to $x_3 = 0$, the lens Ω is the open set of the points lying between Σ^+ and Σ^- whereas the edge is*

$$E = \overline{\Sigma^+} \cap \overline{\Sigma^-}. \tag{3.3.1}$$

We define $\alpha(\mathbf{x})$ as the opening angle between Σ^- and Σ^+ at the point $\mathbf{x} \in E$. We assume that $\alpha(\mathbf{x}) \in (0,\pi)$ for all $\mathbf{x} \in E$.

In our situation the magnetic field $\mathbf{B} = (0,0,1)$ is normal to the plane where the edge lies. For $\mathbf{x} \in \partial\Omega \setminus E$ we introduce the angle $\theta(\mathbf{x})$ defined by:

$$\mathbf{B}\cdot\boldsymbol{n}(\mathbf{x}) = \sin\theta(\mathbf{x}). \tag{3.3.2}$$

A model lens with constant opening angle is given by two parts of a sphere glued together (see Figure 1). In this case we have

$$\forall \mathbf{x} \in \partial\Omega \setminus E, \quad \frac{\pi - \alpha}{2} < \theta(\mathbf{x}) \tag{3.3.3}$$

where $\alpha \in (0, \pi)$ is the opening angle of the lens and we notice that the magnetic field is nowhere tangent to the boundary. We will assume that the opening angle of the lens is variable. For a given point $\mathbf{x}$ of the boundary, we analyze the localized (in a neighborhood of $\mathbf{x}$) magnetic Laplacian $\mathfrak{L}_h^{\mathsf{lens}}$ and we distinguish between $\mathbf{x}$ belonging to the edge and $\mathbf{x}$ belonging to the smooth part of the boundary.

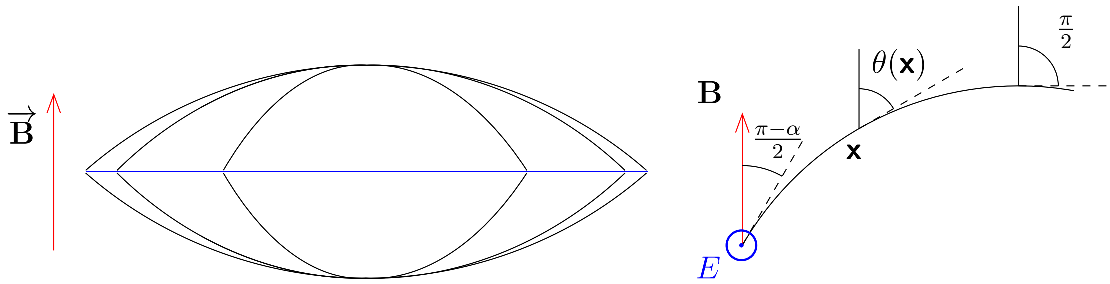


FIGURE 1. A lens Ω: the magnetic field is nowhere tangent to the boundary and it makes the angle $\theta(\mathbf{x})$ with the regular boundary.

3.1.2. *Leading Operator.* Let $\mathbf{x} \in E$ and V a small neighborhood of $\mathbf{x}$ in Ω. We suppose that the opening angle at $\mathbf{x}$ is α. There is a diffeomorphism, denoted by the local coordinates $(\check{s}, \check{t}, \check{z})$, from V to an open subset of the infinite wedge $\mathcal{W}_\alpha$. This diffeomorphism can be explicitly described. We refer to Chapter 2, Section 3.3.3 where some basic properties of the magnetic wedge were discussed.

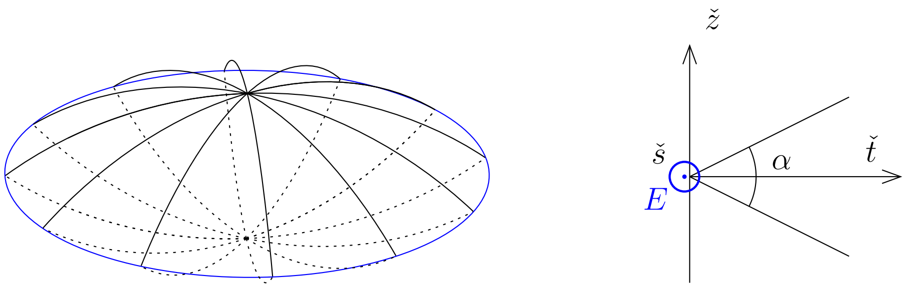


FIGURE 2. Using the local coordinates $(\check{s}, \check{t}, \check{z})$, a neighborhood of a point of the edge can be described as a subset of the infinite wedge $\mathcal{W}_\alpha$.

The model situations (magnetic wedge and smooth boundary) lead to compare the following quantities:

$$\inf_{\mathbf{x}\in E} \nu_1^{\mathsf{e}}(\alpha(\mathbf{x})), \qquad \inf_{\mathbf{x}\in\partial\Omega\setminus E} \mathfrak{s}_1(\theta(\mathbf{x})),$$

where $\theta(\mathbf{x})$ is defined in (3.3.2), $\alpha(\mathbf{x})$ and E are defined in Definition 3.9. Let us state the different assumptions under which we work:

Assumption 3.10.

$$\inf_{\mathbf{x}\in E} \nu_1^{\mathsf{e}}(\alpha(\mathbf{x})) < \inf_{\mathbf{x}\in\partial\Omega\setminus E} \mathfrak{s}_1(\theta(\mathbf{x})). \tag{3.3.4}$$

Remark 3.11. *Using (3.3.3), the fact that $\mathfrak{s}_1$ is increasing and Proposition 2.48, we check that, in the model case when Ω is made of two parts of a sphere glued together, Assumption 3.10 is satisfied for α small enough. By a continuity argument, Assumption 3.10 holds for not too large perturbations of this lens.*

From the properties of the leading operator we see that we will be led to work near the point of the edge of maximal opening. Therefore we will assume the following generic assumption:

Assumption 3.12. *We denote by $\alpha : E \mapsto (0,\pi)$ the opening angle of the lens. We assume that α admits a unique and non degenerate maximum at the point $\mathbf{x}_0$ and we let*

$$\alpha_0 = \max_E \alpha.$$

We denote $\mathcal{T} = \tan\frac{\alpha}{2}$ and $\mathcal{T}_0 = \tan\frac{\alpha_0}{2}$.

In particular, under this assumption and Conjecture 2.50, the function $s \mapsto \nu_1^{\mathsf{e}}(\alpha(s))$ admits a unique and non-degenerate minimum.

3.2. Normal form. This is "classical" that Assumption 3.10 leads to localization properties of the eigenfunctions near the edge E and more precisely near the points of the edge where $E \ni \mathbf{x} \mapsto \nu(\alpha(\mathbf{x}))$ is minimal. Therefore, since ν is decreasing and thanks to Assumption 3.12, we expect that the first eigenfunctions concentrate near the point $\mathbf{x}_0$ where the opening is maximal. This is possible to introduce, near each $\mathbf{x} \in E$, a local change of variables which transforms a neighborhood of $\mathbf{x}$ in Ω in a ε_0-neighborhood of $(0,0,0)$ of $\mathcal{W}_{\alpha(\mathbf{x})}$, denoted by $\mathcal{W}_{\alpha(\mathbf{x}),\varepsilon_0}$.

For the convenience of the reader, let us write below the expression of the magnetic Laplacian in the new local coordinates $(\check{s}, \check{t}, \check{z})$ where $\check{s}$ is a curvilinear abscissa of the edge. The magnetic Laplacian $\mathfrak{L}_h^{\mathsf{lens}}$ is given by the Laplace-Beltrami expression (on $\mathsf{L}^2(|\check{G}|^{1/2}\,\mathrm{d}\check{s}\,\mathrm{d}\check{t}\,\mathrm{d}\check{z})$):

$$\check{\mathfrak{L}}_h^{\mathsf{lens}} := |\check{G}|^{-1/2}\check{\nabla}_h|\check{G}|^{1/2}\check{G}^{-1}\check{\nabla}_h \tag{3.3.5}$$

where:

$$\check{\nabla}_h = \begin{pmatrix} hD_{\check{s}} \\ hD_{\check{t}} \\ h\mathcal{T}(\check{s})^{-1}\mathcal{T}(0)D_{\check{z}} \end{pmatrix} + \begin{pmatrix} -\check{t} + \zeta_0^{\mathsf{e}}h^{1/2} - h\frac{\mathcal{T}'}{2\mathcal{T}}(\check{z}D_{\check{z}} + D_{\check{z}}\check{z}) + \check{R}_1(\check{s},\check{t},\check{z}) \\ 0 \\ 0 \end{pmatrix}. \tag{3.3.6}$$

The precise forms of the Taylor expansions of the remainder $\check{R}_1$, the metric $\check{G}$ and the function $\check{s} \mapsto \mathcal{T}(\check{s})$ are analyzed in [PR13].

Remark 3.13. *Such a normal form allows us to describe the leading structure of this magnetic Laplace-Beltrami operator. Indeed, if we just keep the main terms in* (3.3.5) *by neglecting formally the geometrical factors, our operator takes the simpler form:*

$$(hD_{\check{s}} - \check{t} + \zeta_0^{\mathsf{e}} h^{1/2})^2 + h^2 D_{\check{t}}^2 + h^2 \mathcal{T}(0)^2 \mathcal{T}(\check{s})^{-2} D_{\check{z}}^2,$$

whose symbol with respect to s is discussed in Chapter 2, Section 3.3.3. Performing another formal Taylor expansion near $\check{s} = 0$, we are led to the following operator:

$$(hD_{\check{s}} - \check{t} + \zeta_0^{\mathsf{e}} h^{1/2})^2 + h^2 D_{\check{t}}^2 + h^2 D_{\check{z}}^2 + ch^2 \check{s}^2 D_{\check{z}}^2,$$

where $c > 0$. Using a scaling, we get a rescaled operator whose first term is the leading operator $\mathfrak{L}_{\alpha_0}^{\mathsf{e}}$ and which allows to construct quasimodes. Moreover this form is suitable to establish microlocalization properties of the eigenfunctions with respect to $D_{\check{s}}$.

3.3. Magnetic wells induced by the variations of a singular geometry. The main result of this section is a complete asymptotic expansion of all the first eigenvalues of $\mathfrak{L}_h^{\mathsf{lens}}$.

Theorem 3.14. *We assume that Conjecture 2.50 is true. We also assume Assumptions 3.10 and 3.12. For all $n \geq 1$ there exists $(\mu_{j,n})_{j\geq 0}$ such that we have:*

$$\lambda_n(h) \underset{h\to 0}{\sim} h \sum_{j\geq 0} \mu_{j,n} h^{j/4}.$$

Moreover, we have:

$$\mu_{0,n} = \nu_1^{\mathsf{e}}(\alpha_0), \quad \mu_{1,n} = 0, \quad \mu_{2,n} = \omega_0 + (2n-1)\sqrt{\kappa \mathcal{T}_0^{-1} \|D_{\hat{z}} u_{\zeta_0^{\mathsf{e}}}^{\mathsf{e}}\|^2 \partial_\zeta^2 \nu_1^{\mathsf{e}}(\alpha_0, \zeta_0^{\mathsf{e}})},$$

where ω_0 and $\kappa > 0$ are geometrical constants.

Remark 3.15. *We observe that, for all $n \geq 1$, $\lambda_n(h)$ is simple for h small enough. This simplicity, jointly with a quasimodes construction, also provides an approximation of the corresponding normalized eigenfunction. Moreover, if α is analytic, by using the WKB analysis of* [**BHR14**] *(see Chapter 2, Section 3.3.3), it is possible to get WKB expansions of the eigenfunctions.*

4. Birkhoff normal form

Sections 1, 2 and 3 are mainly structured around the idea of normal forms. Indeed, in each case we have introduced an appropriate change of variable or equivalently a Fourier integral operator and we have *normalized* the magnetic Laplacian by transferring the magnetic geometry into the coefficients of the operator. We can interpret this normalization as a very explicit application of the Egorov theorem. Then, in the investigation, we are led to use the Feshbach projection to simplify again the situation. This projection method can also be heuristically interpreted as a normal form in the spirit of Egorov: taking the average of the operator in a certain quantum state is nothing but the quantum analog of averaging a full Hamiltonian with respect to a reduced Hamiltonian. In problems with boundaries or with vanishing magnetic fields it appears that the dynamics of the reduced Hamiltonian is less understood (due to the boundary for instance) than the spectral theory of its quantization. Keeping this remark in mind it now naturally appears that we should implement a general normal form, for instance in the simplest situation of dimension two, without boundary and with a non vanishing magnetic field. This was the purpose of the collaboration with S. Vũ Ngọc [RVN14].

4.1. Preliminary considerations. As we shall recall below, a particle in a magnetic field has a fast rotating motion, coupled to a slow drift. It is of course expected that the long-time behaviour of the particle is governed by this drift. From the quantum point of view we will see that this drift is governed by a reduced Hamiltonian which can be approximated by the magnetic field itself.

Let (e_1, e_2, e_3) be an orthonormal basis of $\mathbb{R}^3$ and let us consider the plane $\mathbb{R}^2 = \{q_1 e_1 + q_2 e_2;\ (q_1, q_2) \in \mathbb{R}^2\}$, and the magnetic field is $\mathbf{B} = B(q_1, q_2) e_3$. For the moment we only assume that $q = (q_1, q_2)$ belongs to an open set Ω where B does not vanish.

With appropriate constants, Newton's equation for the particle under the action of the Lorentz force writes

$$\ddot{q} = 2\dot{q} \times \mathbf{B}. \tag{3.4.1}$$

The kinetic energy $E = \frac{1}{4}\|\dot{q}\|^2$ is conserved. If the speed $\dot{q}$ is small, we may linearize the system, which amounts to have a constant magnetic field. Then, as is well known, the integration of Newton's equations gives a circular motion of angular velocity $\dot{\theta} = -2B$ and radius $\|\dot{q}\|/2B$. Thus, even if the norm of the speed is small, the angular velocity may be very important. Now, if B is in fact not constant, the particle may leave the region where the linearization is meaningful. This suggests a separation of scales (as in the semiclassical and quantum context of Sections 1 and 3), where the fast circular motion is superposed with a slow motion of the center.

It is known that the system (3.4.1) is Hamiltonian. Let $\mathbf{A} \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$ such that

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

As usual we may identify $\mathbf{A} = (A_1, A_2)$ with the 1-form $A = A_1\, \mathrm{d}q_1 + A_2\, \mathrm{d}q_2$. Then, as a differential 2-form, $\mathrm{d}A = (\frac{\partial A_2}{\partial q_1} - \frac{\partial A_1}{\partial q_2})\, \mathrm{d}q_1 \wedge \mathrm{d}q_2 = B\, \mathrm{d}q_1 \wedge \mathrm{d}q_2$. In terms of canonical variables $(q, p) \in T^*\mathbb{R}^2 = \mathbb{R}^4$ the Hamiltonian of our system is

$$H(q, p) = \|p - \mathbf{A}(q)\|^2. \tag{3.4.2}$$

We use here the Euclidean norm on $\mathbb{R}^2$, which allows the identification of $\mathbb{R}^2$ with $(\mathbb{R}^2)^*$ by

$$\forall (v, p) \in \mathbb{R}^2 \times (\mathbb{R}^2)^*, \qquad p(v) = \langle p, v\rangle. \tag{3.4.3}$$

Thus, the canonical symplectic form ω on $T^*\mathbb{R}^2$ is given by

$$\omega((Q_1, P_1), (Q_2, P_2)) = \langle P_1, Q_2\rangle - \langle P_2, Q_1\rangle. \tag{3.4.4}$$

It is easy to check that Hamilton's equations for H imply Newton's equation (3.4.1). In particular, through the identification (3.4.3) we have $\dot{q} = 2(p - \mathbf{A})$.

4.2. Classical magnetic normal forms. Before considering the semiclassical magnetic Laplacian we shall briefly discuss some results concerning the classical dynamics for large time. As we have already suggested in the introduction of this dissertation, the large time dynamics problem has to face the issue that the conservation of the energy H is not enough to confine the trajectories in a compact set.

The first result shows the existence of a smooth symplectic diffeomorphism that transforms the initial Hamiltonian into a normal form, up to any order in the distance to the zero energy surface.

Theorem 3.16. *Let*

$$H(q,p) := \|p - \boldsymbol{A}(q)\|^2, \quad (q,p) \in T^*\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}^2,$$

where the magnetic potential $\boldsymbol{A} : \mathbb{R}^2 \to \mathbb{R}^2$ is smooth. Let $B := \frac{\partial A_2}{\partial q_1} - \frac{\partial A_1}{\partial q_2}$ be the corresponding magnetic field. Let $\Omega \subset \mathbb{R}^2$ be a bounded open set where B does not vanish. Then there exists a symplectic diffeomorphism Φ, defined in an open set $\tilde{\Omega} \subset \mathbb{C}_{z_1} \times \mathbb{R}^2_{z_2}$, with values in $T^*\mathbb{R}^2$, which sends the plane $\{z_1 = 0\}$ to the surface $\{H = 0\}$, and such that

$$H \circ \Phi = |z_1|^2 f(z_2, |z_1|^2) + \mathcal{O}(|z_1|^\infty), \tag{3.4.5}$$

where $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is smooth. Moreover, the map

$$\varphi : \Omega \ni q \mapsto \Phi^{-1}(q, \boldsymbol{A}(q)) \in (\{0\} \times \mathbb{R}^2_{z_2}) \cap \tilde{\Omega} \tag{3.4.6}$$

is a local diffeomorphism and

$$f \circ (\varphi(q), 0) = |B(q)|.$$

In the following theorem we denote by $K = |z_1|^2 f(z_2, |z_1|^2) \circ \Phi^{-1}$ the (completely integrable) normal form of H given be Theorem 3.16 above. Let φ_H^t be the Hamiltonian flow of H, and let φ_K^t be the Hamiltonian flow of K. Let us state the important dynamical consequences of Theorem 3.16 (see Figure 3).

Theorem 3.17. *Assume that the magnetic field $B > 0$ is confining: there exists $C > 0$ and $M > 0$ such that $B(q) \geq C$ if $\|q\| \geq M$. Let $C_0 < C$. Then*

(1) The flow φ_H^t is uniformly bounded for all starting points (q,p) such that $B(q) \leq C_0$ and $H(q,p) = \mathcal{O}(\epsilon)$ and for times of order $\mathcal{O}(1/\epsilon^N)$, where N is arbitrary.

(2) Up to a time of order $T_\epsilon = \mathcal{O}(|\ln \epsilon|)$, we have

$$\left\|\varphi_H^t(q,p) - \varphi_K^t(q,p)\right\| = \mathcal{O}(\epsilon^\infty) \tag{3.4.7}$$

for all starting points (q,p) such that $B(q) \leq C_0$ and $H(q,p) = \mathcal{O}(\epsilon)$.

It is interesting to notice that, if one restricts to regular values of B, one obtains the same control for a much longer time, as stated below.

Theorem 3.18. *Under the same confinement hypothesis as Theorem 3.17, let $J \subset (0, C_0)$ be a closed interval such that dB does not vanish on $B^{-1}(J)$. Then up to a time of order $T = \mathcal{O}(1/\epsilon^N)$, for an arbitrary $N > 0$, we have*

$$\left\|\varphi_H^t(q,p) - \varphi_K^t(q,p)\right\| = \mathcal{O}(\epsilon^\infty)$$

for all starting points (q,p) such that $B(q) \in J$ and $H(q,p) = \mathcal{O}(\epsilon)$.

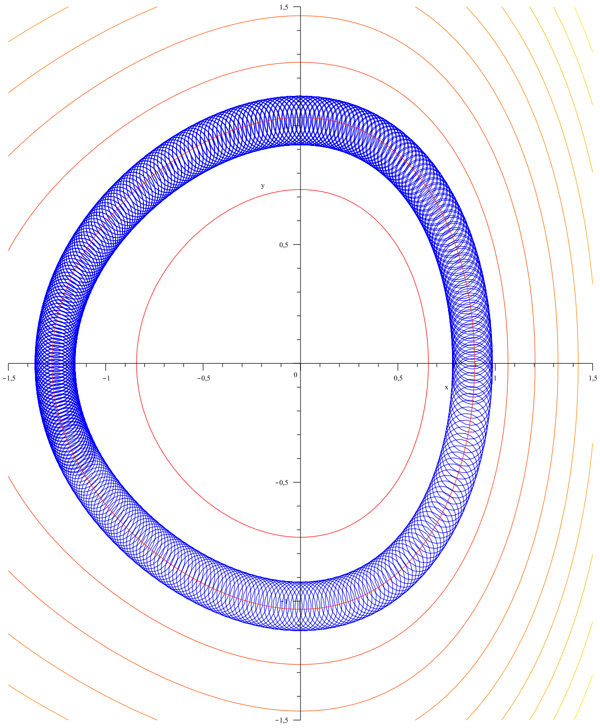


FIGURE 3. Numerical simulation of the flow of H when the magnetic field is given by $B(x,y) = 2 + x^2 + y^2 + \frac{x^3}{3} + \frac{x^4}{20}$, and $\epsilon = 0.05$, $t \in [0, 500]$. The picture also displays in red some level sets of B.

4.3. Semiclassical magnetic normal forms. We turn now to the quantum counterpart of these results. Let $\mathcal{L}_{h,\mathbf{A}} = (-ih\nabla - \mathbf{A})^2$ be the magnetic Laplacian on $\mathbb{R}^2$, where the potential $\mathbf{A} : \mathbb{R}^2 \to \mathbb{R}^2$ is smooth, and such that $\mathcal{L}_{h,\mathbf{A}} \in S(m)$ for some order function m on $\mathbb{R}^4$ (see [35, Chapter 7]). We will work with the Weyl quantization; for a classical symbol $a = a(x,\xi) \in S(m)$, it is defined as:

$$\mathsf{Op}_h^w a\, \psi(x) = \frac{1}{(2\pi h)^2} \int\int e^{i(x-y)\cdot\xi/h} a\left(\frac{x+y}{2}, \xi\right) \psi(y)\, \mathrm{d}y\, \mathrm{d}\xi, \quad \forall \psi \in \mathcal{S}(\mathbb{R}^2).$$

The first result shows that the spectral theory of $\mathcal{L}_{h,\mathbf{A}}$ is governed at first order by the magnetic field itself, viewed as a symbol.

Theorem 3.19. *Assume that the magnetic field B is non vanishing on $\mathbb{R}^2$ and confining: there exist constants $\tilde{C}_1 > 0$, $M_0 > 0$ such that*

$$B(q) \geq \tilde{C}_1 \quad \text{for} \quad |q| \geq M_0. \tag{3.4.8}$$

Let $\mathcal{H}_h^0 = \mathsf{Op}_h^w(H^0)$, where $H^0 = B(\varphi^{-1}(z_2))|z_1|^2$ where $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ is a diffeomorphism. Then there exists a bounded classical pseudo-differential operator Q_h on $\mathbb{R}^2$, such that

- *Q_h commutes with $\mathsf{Op}_h^w(|z_1|^2)$;*
- *Q_h is relatively bounded with respect to $\mathcal{H}_h^0$ with an arbitrarily small relative bound;*
- *its Weyl symbol is $O_{z_2}(h^2 + h|z_1|^2 + |z_1|^4)$,*

so that the following holds. Let $0 < C_1 < \tilde{C}_1$. Then the spectra of $\mathcal{L}_{h,\boldsymbol{A}}$ and $\mathcal{L}_h^{\mathsf{No}} := \mathcal{H}_h^0 + Q_h$ in $(-\infty, C_1 h]$ are discrete. We denote by $0 < \lambda_1(h) \leq \lambda_2(h) \leq \cdots$ the eigenvalues of $\mathcal{L}_{h,\boldsymbol{A}}$ and by $0 < \mu_1(h) \leq \mu_2(h) \leq \cdots$ the eigenvalues of $\mathcal{L}_h^{\mathsf{No}}$. Then for all $j \in \mathbb{N}^*$ such that $\lambda_j(h) \leq C_1 h$ and $\mu_j(h) \leq C_1 h$, we have

$$|\lambda_j(h) - \mu_j(h)| = O(h^\infty).$$

The proof of Theorem 3.19 relies on the following theorem (see [86] where a close form of this theorem appears), which provides in particular an accurate description of Q_h. In the statement, we use the notation of Theorem 3.16. We recall that Σ is the zero set of the classical Hamiltonian H.

Theorem 3.20. *For h small enough there exists a Fourier Integral Operator U_h such that*

$$U_h^* U_h = I + Z_h, \qquad U_h U_h^* = I + Z_h',$$

where Z_h, Z_h' are pseudo-differential operators that microlocally vanish in a neighborhood of $\tilde{\Omega} \cap \Sigma$, and

$$U_h^* \mathcal{L}_{h,\boldsymbol{A}} U_h = \mathcal{L}_h^{\mathsf{No}} + R_h, \tag{3.4.9}$$

where

(1) $\mathcal{L}_h^{\mathsf{No}}$ is a classical pseudo-differential operator in $S(m)$ that commutes with

$$\mathcal{I}_h := -h^2 \frac{\partial^2}{\partial x_1^2} + x_1^2;$$

(2) For any Hermite function $h_n(x_1)$ such that $\mathcal{I}_h h_n = h(2n-1)h_n$, the operator $\mathcal{L}_h^{\mathsf{No},(n)}$ acting on $L^2(\mathbb{R}_{x_2})$ by

$$h_n \otimes \mathcal{L}_h^{\mathsf{No},(n)}(u) = \mathcal{L}_h^{\mathsf{No}}(h_n \otimes u)$$

is a classical pseudo-differential operator in $S_{\mathbb{R}^2}(m)$ of h-order 1 with principal symbol

$$F^{(n)}(x_2, \xi_2) = h(2n-1)B(q),$$

where $(0, x_2 + i\xi_2) = \varphi(q)$ as in (3.4.6);

(3) *Given any classical pseudo-differential operator D_h with principal symbol d_0 such that $d_0(z_1, z_2) = c(z_2)|z_1|^2 + O(|z_1|^3)$, and any $N \geq 1$, there exist classical pseudo-differential operators $S_{h,N}$ and K_N such that:*

$$R_h = S_{h,N}(D_h)^N + K_N + O(h^\infty), \tag{3.4.10}$$

with K_N compactly supported away from a fixed neighborhood of $|z_1| = 0$.

(4) $\mathcal{L}_h^{\mathsf{No}} = \mathcal{H}_h^0 + Q_h$, *where $\mathcal{H}_h^0 = \mathsf{Op}_h^w(H^0)$, $H^0 = B(\varphi^{-1}(z_2))|z_1|^2$, and the operator Q_h is relatively bounded with respect to $\mathcal{H}_h^0$ with an arbitrarily small relative bound.*

We recover the result of [69], adding the fact that no odd power of $h^{1/2}$ can show up in the asymptotic expansion (see the recent work [72] where a Grushin type method is used to obtain a close result).

Corollary 3.21 (Low lying eigenvalues). *Assume that B has a unique non-degenerate minimum. Then there exists a constant c_0 such that for any j, the eigenvalue $\lambda_j(h)$ has a full asymptotic expansion in integral powers of h whose first terms have the following form:*

$$\lambda_j(h) \sim h \min B + h^2(c_1(2j-1) + c_0) + O(h^3),$$

with $c_1 = \frac{\sqrt{\det(B" \circ \varphi^{-1}(0))}}{2B \circ \varphi^{-1}(0)}$, where the minimum of B is reached at $\varphi^{-1}(0)$.

Proof. The first eigenvalues of $\mathcal{L}_{h,A}$ are equal to the eigenvalues of $\mathcal{L}_h^{\mathsf{No},(1)}$ (in point (2) of Theorem 3.20). Since B has a non-degenerate minimum, the symbol of $\mathcal{L}_h^{\mathsf{No},(1)}$ has a non-degenerate minimum, and the spectral asymptotics of the low-lying eigenvalues for such a 1D pseudo-differential operator are well known. We get

$$\lambda_j(h) \sim h \min B + h^2(c_1(2j-1) + c_0) + O(h^3),$$

with $c_1 = \sqrt{\det(B \circ \varphi^{-1})"(0)/2}$. One can easily compute

$$c_1 = \frac{\sqrt{\det(B" \circ \varphi^{-1}(0))}}{2\,|\det(D\varphi^{-1}(0))|} = \frac{\sqrt{\det(B" \circ \varphi^{-1}(0))}}{2B \circ \varphi^{-1}(0)}.$$

□

CHAPTER 4

Waveguides

Si on me presse de dire pourquoi je l'aimais, je sens que cela ne se peut exprimer qu'en répondant : Parce que c'était lui : parce que c'était moi.

Les Essais, Livre I, Chapitre XXVIII, Montaigne

This chapter presents recent results in the spectral theory of waveguides. It is essentially based on the collaborations with D. Krejčiřík [KR13] on the one hand and with M. Dauge [DaR12] on the other hand. In Section 1 we describe magnetic waveguides in dimensions two and three and we analyze the spectral influence of the width ε of the waveguide and the intensity b if the magnetic field. In particular we investigate the limit $\varepsilon \to 0$. In Section 2 we describe the same problem in the case of layers. In Sections 3 and 4 the effect of a corner in dimension two is tackled.

1. Magnetic waveguides

This section is concerned with spectral properties of a curved quantum waveguide when a magnetic field is applied. The main results of this section were obtained in collaboration with D. Krejčiřík in [KR13].

We will give a precise definition of what a waveguide is in Sections 1.3 and 1.4. Without going into the details we can already mention that we will use the definition given in the famous (non magnetic) paper of Duclos and Exner [37] and its generalizations [26, 93, 56]. The waveguide is nothing but a tube Ω_ε about an unbounded curve γ in the Euclidean space $\mathbb{R}^d$, with $d \geq 2$, where ε is a positive shrinking parameter and the cross section is defined as $\varepsilon\omega = \{\varepsilon\tau : \tau \in \omega\}$.

More precisely this section is devoted to the spectral analysis of the magnetic operator with Dirichlet boundary conditions $\mathfrak{L}^{[d]}_{\varepsilon,b\mathbf{A}}$ defined as

$$(-i\nabla_x + b\,\mathbf{A}(x))^2 \qquad \text{on} \qquad \mathsf{L}^2(\Omega_\varepsilon,\, \mathrm{d}x). \tag{4.1.1}$$

where $b > 0$ is a positive parameter and $\mathbf{A}$ a smooth vector potential associated with a given magnetic field $\mathbf{B}$.

1.1. The result of Duclos and Exner. One of the remarkable facts which is proved by Duclos and Exner is that the Dirichlet Laplacian on Ω_ε always has discrete spectrum below its essential spectrum when the waveguide is not straight and asymptotically straight. Let us sketch the proof of this result in the case of two dimensional waveguides.

Let us consider a smooth and injective curve $\gamma\colon \mathbb{R} \ni s \mapsto \gamma(s)$ which is parameterized by its arc length s. The normal to the curve at $\gamma(s)$ is defined as the unique unit vector $\boldsymbol{n}(s)$ such that $\gamma'(s) \cdot \nu(s) = 0$ and $\det(\gamma', \nu) = 1$. We have the relation $\gamma''(s) = \kappa(s)\boldsymbol{n}(s)$ where $\kappa(s)$ denotes the algebraic curvature at the point $\gamma(s)$. We can now define standard tubular coordinates. We consider:

$$\mathbb{R} \times (-\varepsilon, \varepsilon) \ni (s,t) \mapsto \Phi(s,t) = \gamma(s) + t\boldsymbol{n}(s).$$

We always assume

$$\Phi \text{ is injective} \qquad \text{and} \qquad \varepsilon \sup_{s\in\mathbb{R}} |\kappa(s)| < 1. \tag{4.1.2}$$

Then it is well known (see [93]) that Φ defines a smooth diffeomorphism from $\mathbb{R} \times (-\varepsilon, \varepsilon)$ onto the image $\Omega_\varepsilon = \Phi(\mathbb{R} \times (-\varepsilon, \varepsilon))$, which we identify with our waveguide. In these new coordinates, the operator becomes (exercise)

$$\mathfrak{L}^{[2]}_{\varepsilon,0} = -m^{-1}\partial_s m^{-1}\partial_s - m^{-1}\partial_t m \partial_t, \quad m(s,t) = 1 - t\kappa(s),$$

which is acting in the weighted space $\mathsf{L}^2(\mathbb{R} \times (-\varepsilon, \varepsilon), m(s,t)\, \mathrm{d}s\, \mathrm{d}t)$. We introduce the shifted quadratic form:

$$\mathcal{Q}^{[2],\mathsf{sh}}_{\varepsilon,0}(\phi) = \int_{\mathbb{R}\times(-\varepsilon,\varepsilon)} \left(m^{-2}|\partial_s(\phi)|^2 + |\partial_t \phi|^2 - \frac{\pi^2}{4\varepsilon^2}|\phi|^2 \right) m \, \mathrm{d}s\, \mathrm{d}t$$

and we let:

$$\phi_n(s,t) = \chi_0(n^{-1}s) \cos\left(\frac{\pi}{2\varepsilon} t\right),$$

where χ_0 is a smooth cutoff function which is 1 near 0. We can check that $\mathcal{Q}^{[2],\mathsf{sh}}_{\varepsilon,0}(\phi_n) \underset{n\to+\infty}{\to} 0$. Let us now consider a smooth cutoff function χ_1 which is 1 near a point where κ is not zero and define $\tilde{\phi}(s,t) = -\chi_1^2(s,t)\mathcal{L}^{[2],\mathsf{sh}}_{\varepsilon,0}\phi_n(s,t)$ which does not depend on n as soon as n is large enough. Then we have:

$$\mathcal{Q}^{[2],\mathsf{sh}}_{\varepsilon,0}(\phi_n + \eta\tilde{\phi}) = \mathcal{Q}^{[2],\mathsf{sh}}_{\varepsilon,0}(\phi_n) - 2\eta\mathcal{B}^{[2],\mathsf{sh}}_{\varepsilon,0}(\phi_n, \chi_1(s)\mathcal{L}^{[2],\mathsf{sh}}_{\varepsilon,0}\phi_n) + \eta^2\mathcal{Q}^{[2],\mathsf{sh}}_{\varepsilon,0}(\tilde{\phi}).$$

For n large enough, the quantity $\mathcal{B}^{[2],\mathsf{sh}}_{\varepsilon,0}(\phi_n, \chi_1(s)\mathcal{L}^{[2],\mathsf{sh}}_{\varepsilon,0}\phi_n)$ does not depend on n and is positive. For such an n, we take η small enough and we find:

$$\mathcal{Q}^{[2],\mathsf{sh}}_{\varepsilon,0}(\phi_n + \eta\tilde{\phi}) < 0.$$

Therefore the bottom of the spectrum is an eigenvalue due to the min-max principle.

Duclos and Exner also investigate the limit $\varepsilon \to 0$ to show that the Dirichlet Laplacian on the tube Ω_ε converges in a suitable sense to the effective one dimensional operator

$$\mathcal{L}^{\mathsf{eff}} = -\partial_s^2 - \frac{\kappa(s)^2}{4} \qquad \text{on} \qquad \mathsf{L}^2(\gamma, \mathrm{d}s).$$

In addition it is proved in [37] that each eigenvalue of this effective operator generates an eigenvalue of the Dirichlet Laplacian on the tube.

As Duclos and Exner we are interested in approximations of $\mathfrak{L}^{[d]}_{\varepsilon,b\mathbf{A}}$ in the small cross section limit $\varepsilon \to 0$. Such an approximation might non trivially depends on the intensity of the magnetic field b especially if it is allowed to depend on ε.

1.2. Waveguides with more geometry. In dimension three it is also possible to twist the waveguide by allowing the cross section of the waveguide to non-trivially rotate by an angle function θ with respect to a relatively parallel frame of γ (then the velocity θ' can be interpreted as a "torsion"). It is proved in [41] that, whereas the curvature is favourable to discrete spectrum, the torsion plays against it. In particular, the spectrum of a straight twisted waveguide is stable under small perturbations (such as local electric field or bending). This repulsive effect of twisting is quantified in [41] (see also [92, 95]) by means of a Hardy type inequality. The limit $\varepsilon \to 0$ permits to compare the effects bending and twisting ([19, 33, 94]) and the effective operator is given by

$$\mathcal{L}^{\mathsf{eff}} = -\partial_s^2 - \frac{\kappa(s)^2}{4} + C(\omega)\theta'(s)^2 \qquad \text{on} \qquad \mathsf{L}^2(\gamma, \mathrm{d}s),$$

where $C(\omega)$ is a positive constant whenever ω is not a disk or annulus. Writing (4.1.1)

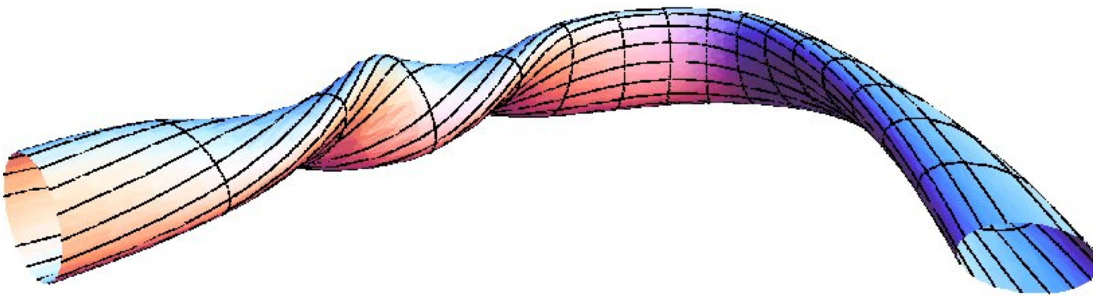


FIGURE 1. Torsion on the left and curvature on the right

in suitable curvilinear coordinates (see (4.1.9) below), one may notice similarities in the appearance of the torsion and the magnetic field in the coefficients of the operator and it therefore seems natural to ask the following question:

"Does the magnetic field act as the torsion ?"

In order to define our effective operators in the limit $\varepsilon \to 0$ we shall describe more accurately the geometry of our waveguides. This is the aim of the next two sections in which we will always assume that the geometry (curvature and twist) and the magnetic field are compactly supported.

1.3. Two-dimensional waveguides. Up to changing the gauge, the Laplace-Beltrami expression of $\mathfrak{L}^{[2]}_{\varepsilon,b\mathbf{A}}$ in these coordinates is given by

$$\mathfrak{L}^{[2]}_{\varepsilon,b\mathcal{A}} = (1-t\kappa(s))^{-1}(i\partial_s+b\mathcal{A}_1)(1-t\kappa(s))^{-1}(i\partial_s+b\mathcal{A}_1)-(1-t\kappa(s))^{-1}\partial_t(1-t\kappa(s))\partial_t,$$

with the gauge:

$$\mathcal{A}(s,t)=(\mathcal{A}_1(s,t),0),\quad \mathcal{A}_1(s,t)=\int_0^t(1-t'\kappa(s))\mathbf{B}(\Phi(s,t'))\,\mathrm{d}t'.$$

We let:

$$m(s,t)=1-t\kappa(s).$$

The self-adjoint operator $\mathfrak{L}^{[2]}_{\varepsilon,b\mathcal{A}}$ on $\mathsf{L}^2(\mathbb{R}\times(-\varepsilon,\varepsilon), m\,\mathrm{d}s\,\mathrm{d}t)$ is unitarily equivalent to the self-adjoint operator on $\mathsf{L}^2(\mathbb{R}\times(-\varepsilon,\varepsilon),\,\mathrm{d}s\,\mathrm{d}t)$:

$$\mathcal{L}^{[2]}_{\varepsilon,b\mathcal{A}} = m^{1/2}\mathfrak{L}^{[2]}_{\varepsilon,b\mathbf{A}}m^{-1/2}.$$

Introducing the rescaling

$$t=\varepsilon\tau, \tag{4.1.3}$$

we let:

$$\mathcal{A}_\varepsilon(s,\tau)=(\mathcal{A}_{1,\varepsilon}(s,\tau),0)=(\mathcal{A}_1(s,\varepsilon\tau),0)$$

and denote by $\mathcal{L}^{[2]}_{\varepsilon,b\mathcal{A}_\varepsilon}$ the homogenized operator on $\mathsf{L}^2(\mathbb{R}\times(-1,1),\,\mathrm{d}s\,\mathrm{d}\tau)$:

$$\mathcal{L}^{[2]}_{\varepsilon,b\mathcal{A}_\varepsilon} = m_\varepsilon^{-1/2}(i\partial_s+b\mathcal{A}_{1,\varepsilon})m_\varepsilon^{-1}(i\partial_s+b\mathcal{A}_{1,\varepsilon})m_\varepsilon^{-1/2}-\varepsilon^{-2}\partial_\tau^2+V_\varepsilon(s,\tau), \tag{4.1.4}$$

with:

$$m_\varepsilon(s,\tau)=m(s,\varepsilon\tau),\quad V_\varepsilon(s,\tau)=-\frac{\kappa(s)^2}{4}(1-\varepsilon\kappa(s)\tau)^{-2}.$$

It is easy to verify that $\mathcal{L}^{[2]}_{\varepsilon,b\mathcal{A}}$, defined as Friedrich extension of the operator initially defined on $\mathcal{C}_0^\infty(\mathbb{R}\times(-\varepsilon,\varepsilon))$, has form domain $\mathsf{H}^1_0(\mathbb{R}\times(-\varepsilon,\varepsilon))$. Similarly, the form domain of $\mathcal{L}^{[2]}_{\varepsilon,b\mathcal{A}_\varepsilon}$ is $\mathsf{H}^1_0(\mathbb{R}\times(-1,1))$.

1.4. Three-dimensional waveguides. The situation is geometrically more complicated in dimension 3. We consider a smooth curve γ which is parameterized by its arc length s and does not overlap itself. We use the so-called Tang frame (or the relatively parallel frame, see for instance [94]) to describe the geometry of the tubular neighbourhood of γ. Denoting the (unit) tangent vector by $T(s)=\gamma'(s)$, the Tang frame $(T(s),M_2(s),M_3(s))$ satisfies the relations:

$$\begin{aligned} T' &= \kappa_2M_2+\kappa_3M_3,\\ M_2' &= -\kappa_2T,\\ M_3' &= -\kappa_3T. \end{aligned}$$

The functions κ_2 and κ_3 are the curvatures related to the choice of the normal fields M_2 and M_3. We can notice that $\kappa^2=\kappa_2^2+\kappa_3^2=|\gamma''|^2$ is the square of the usual curvature of γ.

Let $\theta : \mathbb{R} \to \mathbb{R}$ a smooth function (twisting). We introduce the map $\Phi : \mathbb{R}\times(\varepsilon\omega) \to \Omega_\varepsilon$ defined by:
(4.1.5)
$$x = \Phi(s, t_2, t_3) = \gamma(s) + t_2(\cos\theta M_2(s) + \sin\theta M_3(s)) + t_3(-\sin\theta M_2(s) + \cos\theta M_3(s)).$$
Let us notice that s will often be denoted by t_1. As in dimension two, we always assume:

(4.1.6) $$\Phi \text{ is injective} \quad \text{and} \quad \varepsilon \sup_{(\tau_2,\tau_3)\in\omega} (|\tau_2| + |\tau_3|) \sup_{s\in\mathbb{R}} |\kappa(s)| < 1.$$

Sufficient conditions ensuring the injectivity hypothesis can be found in [41, App. A]. We define $\mathcal{A} = D\Phi\mathbf{A}(\Phi) = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$,

$$\begin{aligned} h &= 1 - t_2(\kappa_2\cos\theta + \kappa_3\sin\theta) - t_3(-\kappa_2\sin\theta + \kappa_3\cos\theta), \\ h_2 &= -t_2\theta', \\ h_3 &= t_3\theta', \end{aligned}$$

and $\mathcal{R} = h_3 b\mathcal{A}_2 + h_2 b\mathcal{A}_3$. We also introduce the angular derivative $\partial_\alpha = t_3\partial_{t_2} - t_2\partial_{t_3}$. The magnetic operator $\mathfrak{L}^{[3]}_{\varepsilon,b\mathbf{A}}$ is unitarily equivalent to the operator on $\mathsf{L}^2(\Omega_\varepsilon, h\,\mathrm{d}t)$ given by

(4.1.7) $$\mathfrak{L}^{[3]}_{\varepsilon,b\mathcal{A}} = \sum_{j=2,3} h^{-1}(-i\partial_{t_j} + b\mathcal{A}_j)h(-i\partial_{t_j} + b\mathcal{A}_j) + h^{-1}(-i\partial_s + b\mathcal{A}_1 - i\theta'\partial_\alpha + \mathcal{R})h^{-1}(-i\partial_s + b\mathcal{A}_1 - i\theta'\partial_\alpha + \mathcal{R}).$$

By considering the conjugate operator $h^{1/2}\mathfrak{L}^{[3]}_{\varepsilon,b\mathcal{A}}h^{-1/2}$, we find that $\mathfrak{L}^{[3]}_{\varepsilon,b\mathcal{A}}$ is unitarily equivalent to the operator defined on $\mathsf{L}^2(\mathbb{R}\times(\varepsilon\omega), \mathrm{d}s\,\mathrm{d}t_2\,\mathrm{d}t_3)$ given by:

(4.1.8) $$\mathcal{L}^{[3]}_{\varepsilon,b\mathcal{A}} = \sum_{j=2,3} (-i\partial_{t_j} + b\mathcal{A}_j)^2 - \frac{\kappa^2}{4h^2} + h^{-1/2}(-i\partial_s + b\mathcal{A}_1 - i\theta'\partial_\alpha + \mathcal{R})h^{-1}(-i\partial_s + b\mathcal{A}_1 - i\theta'\partial_\alpha + \mathcal{R})h^{-1/2}.$$

Finally, introducing the rescaling

$$(t_2, t_3) = \varepsilon(\tau_2, \tau_3) = \varepsilon\tau,$$

we define the homogenized operator on $\mathsf{L}^2(\mathbb{R}\times\omega, \mathrm{d}s\,\mathrm{d}\tau)$:

(4.1.9) $$\mathcal{L}^{[3]}_{\varepsilon,b\mathcal{A}_\varepsilon} = \sum_{j=2,3} (-i\varepsilon^{-1}\partial_{\tau_j} + b\mathcal{A}_{j,\varepsilon})^2 - \frac{\kappa^2}{4h_\varepsilon^2} + h_\varepsilon^{-1/2}(-i\partial_s + b\mathcal{A}_{1,\varepsilon} - i\theta'\partial_\alpha + \mathcal{R}_\varepsilon)h_\varepsilon^{-1}(-i\partial_s + b\mathcal{A}_{1,\varepsilon} - i\theta'\partial_\alpha + \mathcal{R}_\varepsilon)h_\varepsilon^{-1/2},$$

where $\mathcal{A}_\varepsilon(s,\tau) = \mathcal{A}(s,\varepsilon\tau)$, $h_\varepsilon(s,\tau) = h(s,\varepsilon\tau)$ and $\mathcal{R}_\varepsilon = \mathcal{R}(s,\varepsilon\tau)$.

The form domains of $\mathcal{L}^{[3]}_{\varepsilon,b\mathcal{A}}$ and $\mathcal{L}^{[3]}_{\varepsilon,b\mathcal{A}_\varepsilon}$ are $\mathsf{H}^1_0(\mathbb{R}\times(-\varepsilon,\varepsilon))$ and $\mathsf{H}^1_0(\mathbb{R}\times(-1,1))$, respectively.

1.5. Limiting models and asymptotic expansions. We can now state our main results concerning the effective models in the limit $\varepsilon \to 0$. We will denote by $\lambda_n^{\mathsf{Dir}}(\omega)$ the n-th eigenvalue of the Dirichlet Laplacian $-\Delta_\omega^{\mathsf{Dir}}$ on $\mathsf{L}^2(\omega)$. The first positive and L^2-normalized eigenfunction will be denoted by J_1.

Definition 4.1 (Case $d = 2$)**.** *For $\delta \in (-\infty, 1)$, we define:*

$$\mathcal{L}_{\varepsilon,\delta}^{\mathsf{eff},[2]} = -\varepsilon^{-2}\Delta_\omega^{\mathsf{Dir}} - \partial_s^2 - \frac{\kappa(s)^2}{4}$$

and for $\delta = 1$, we let:

$$\mathcal{L}_{\varepsilon,1}^{\mathsf{eff},[2]} = -\varepsilon^{-2}\Delta_\omega^{\mathsf{Dir}} + \mathcal{T}^{[2]},$$

where

$$\mathcal{T}^{[2]} = -\partial_s^2 + \left(\frac{1}{3} + \frac{2}{\pi^2}\right)\boldsymbol{B}(\gamma(s))^2 - \frac{\kappa(s)^2}{4}.$$

Theorem 4.2 (Case $d = 2$)**.** *There exists K such that, for all $\delta \in (-\infty, 1]$, there exist $\varepsilon_0 > 0, C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$:*

$$\left\|\left(\mathcal{L}^{[2]}_{\varepsilon,\varepsilon^{-\delta}\mathcal{A}_\varepsilon} - \varepsilon^{-2}\lambda_1^{\mathsf{Dir}}(\omega) + K\right)^{-1} - \left(\mathcal{L}^{\mathsf{eff},[2]}_{\varepsilon,\delta} - \varepsilon^{-2}\lambda_1^{\mathsf{Dir}}(\omega) + K\right)^{-1}\right\| \leq C\max(\varepsilon^{1-\delta}, \varepsilon), \text{ for } \delta < 1$$

and:

$$\left\|\left(\mathcal{L}^{[2]}_{\varepsilon,\varepsilon^{-1}\mathcal{A}_\varepsilon} - \varepsilon^{-2}\lambda_1^{\mathsf{Dir}}(\omega) + K\right)^{-1} - \left(\mathcal{L}^{\mathsf{eff},[2]}_{\varepsilon,1} - \varepsilon^{-2}\lambda_1^{\mathsf{Dir}}(\omega) + K\right)^{-1}\right\| \leq C\varepsilon.$$

In the critical regime $\delta = 1$, we deduce the following corollary providing the asymptotic expansions of the lowest eigenvalues $\lambda_n^{[2]}(\varepsilon)$ of $\mathcal{L}^{[2]}_{\varepsilon,\varepsilon^{-1}\mathcal{A}_\varepsilon}$.

Corollary 4.3 (Case $d = 2$ and $\delta = 1$)**.** *Let us assume that $\mathcal{T}^{[2]}$ admits N (simple) eigenvalues $\mu_0, \cdots, \mu_N$ below the threshold of the essential spectrum. Then, for all $n \in \{1, \cdots N\}$, there exist $(\delta_{j,n})_{j\geq 0}$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$:*

$$\lambda_n^{[2]}(\varepsilon) \underset{\varepsilon\to 0}{\sim} \sum_{j\geq 0} \delta_{j,n}\varepsilon^{-2+j},$$

with

$$\delta_{0,n} = \frac{\pi^2}{4}, \quad \delta_{1,n} = 0, \quad \delta_{2,n} = \mu_n.$$

Thanks to the spectral theorem, we also get the approximation of the corresponding eigenfunctions at any order.

In order to present analogous results in dimension three, we introduce supplementary notation. The norm and the inner product in $\mathsf{L}^2(\omega)$ will be denoted by $\|\cdot\|_\omega$ and $\langle\cdot,\cdot\rangle_\omega$, respectively.

Definition 4.4 (Case $d = 3$)**.** *For $\delta \in (-\infty, 1)$, we define:*

$$\mathcal{L}_{\varepsilon,\delta}^{\mathsf{eff},[3]} = -\varepsilon^{-2}\Delta_\omega^{\mathsf{Dir}} - \partial_s^2 - \frac{\kappa(s)^2}{4} + \|\partial_\alpha J_1\|_\omega^2\theta'^2$$

and for $\delta = 1$, we let:

$$\mathcal{L}^{\mathsf{eff},[3]}_{\varepsilon,1} = -\varepsilon^{-2}\Delta^{\mathsf{Dir}}_{\omega} + \mathcal{T}^{[3]},$$

where $\mathcal{T}^{[3]}$ is defined by:

$$\begin{aligned}\mathcal{T}^{[3]} = \langle(-i\partial_s - i\theta'\partial_\alpha - \mathcal{B}_{12}(s,0,0)\tau_2 - \mathcal{B}_{13}(s,0,0)\tau_3)^2 \mathsf{Id}(s)\otimes J_1, \mathsf{Id}(s)\otimes J_1\rangle_\omega \\ + \mathcal{B}_{23}^2(s,0,0)\left(\frac{\|\tau J_1\|_\omega^2}{4} - \langle D_\alpha R_\omega, J_1\rangle_\omega\right) - \frac{\kappa^2(s)}{4},\end{aligned}$$

*where R_ω is a determined function (see [**KR13**]) and*

$$\begin{aligned}\mathcal{B}_{23}(s,0,0) &= \boldsymbol{B}(\gamma(s))\cdot T(s),\\ \mathcal{B}_{13}(s,0,0) &= \boldsymbol{B}(\gamma(s))\cdot(\cos\theta\, M_2(s) - \sin\theta\, M_3(s)),\\ \mathcal{B}_{12}(s,0,0) &= \boldsymbol{B}(\gamma(s))\cdot(-\sin\theta\, M_2(s) + \cos\theta\, M_3(s)).\end{aligned}$$

Theorem 4.5 (Case $d = 3$)**.** *There exists K such that for all $\delta \in (-\infty, 1]$, there exist $\varepsilon_0 > 0, C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$:*

$$\left\|\left(\mathcal{L}^{[3]}_{\varepsilon,\varepsilon^{-\delta}\mathcal{A}_\varepsilon} - \varepsilon^{-2}\lambda_1^{\mathsf{Dir}}(\omega) + K\right)^{-1} - \left(\mathcal{L}^{\mathsf{eff},[3]}_{\varepsilon,\delta} - \varepsilon^{-2}\lambda_1^{\mathsf{Dir}}(\omega) + K\right)^{-1}\right\| \leq C\max(\varepsilon^{1-\delta},\varepsilon),\ \textit{for } \delta < 1$$

and:

$$\left\|\left(\mathcal{L}^{[3]}_{\varepsilon,\varepsilon^{-1}\mathcal{A}_\varepsilon} - \varepsilon^{-2}\lambda_1^{\mathsf{Dir}}(\omega) + K\right)^{-1} - \left(\mathcal{L}^{\mathsf{eff},[3]}_{\varepsilon,1} - \varepsilon^{-2}\lambda_1^{\mathsf{Dir}}(\omega) + K\right)^{-1}\right\| \leq C\varepsilon.$$

In the same way, this theorem implies asymptotic expansions of eigenvalues $\lambda_n^{[3]}(\varepsilon)$ of $\mathcal{L}^{[3]}_{\varepsilon,\varepsilon^{-1}\mathcal{A}_\varepsilon}$.

Corollary 4.6 (Case $d = 3$ and $\delta = 1$)**.** *Let us assume that $\mathcal{T}^{[3]}$ admits N (simple) eigenvalues $\nu_0, \cdots, \nu_N$ below the threshold of the essential spectrum. Then, for all $n \in \{1, \cdots N\}$, there exist $(\delta_{j,n})_{j\geq 0}$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$:*

$$\lambda_n^{[3]}(\varepsilon) \underset{\varepsilon\to 0}{\sim} \sum_{j\geq 0} \delta_{j,n}\varepsilon^{-2+j},$$

with

$$\delta_{0,n} = \lambda_1^{\mathsf{Dir}}(\omega), \quad \delta_{1,n} = 0, \quad \delta_{2,n} = \nu_n.$$

As in two dimensions, we also get the corresponding expansion for the eigenfunctions. Complete asymptotic expansions for eigenvalues in finite three-dimensional waveguides without magnetic field are also previously established in [**63, 15**]. Such expansions were also obtained in [**62**] in the case $\delta = 0$ in a periodic framework.

Remark 4.7. *As expected, when $\delta = 0$ that is when b is kept fixed, the magnetic field does not persists in the limit $\varepsilon \to 0$ as well in dimension two as in dimension three. Indeed, in this limit Ω_ε converges to the one dimensional curve γ and there is no magnetic field in dimension 1.*

1.6. Norm resolvent convergence. Let us state an auxiliary result, inspired by the approach of [58], which tells us that, in order to estimate the difference between two resolvents, it is sufficient to analyse the difference between the corresponding sesquilinear forms as soon as their domains are the same.

Lemma 4.8. *Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be two positive self-adjoint operators on a Hilbert space* H*. Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be their associated sesquilinear forms. We assume that* $\mathsf{Dom}\,(\mathfrak{B}_1) = \mathsf{Dom}\,(\mathfrak{B}_2)$*. Assume that there exists $\eta > 0$ such that for all $\phi, \psi \in \mathsf{Dom}\,(\mathfrak{B}_1)$:*

$$|\mathfrak{B}_1(\phi,\psi) - \mathfrak{B}_2(\phi,\psi)| \leq \eta\sqrt{\mathfrak{Q}_1(\psi)}\sqrt{\mathfrak{Q}_2(\phi)},$$

where $\mathfrak{Q}_j(\varphi) = \mathfrak{B}_j(\varphi,\varphi)$ for $j = 1,2$ and $\varphi \in \mathsf{Dom}\,(\mathfrak{B}_1)$. Then, we have:

$$\|\mathfrak{L}_1^{-1} - \mathfrak{L}_2^{-1}\| \leq \eta\|\mathfrak{L}_1^{-1}\|^{1/2}\|\mathfrak{L}_2^{-1}\|^{1/2}.$$

Proof. The original proof can be found in [94, Prop. 5.3]. Let us consider $\tilde{\phi}, \tilde{\psi} \in \mathsf{H}$. We let $\phi = \mathfrak{L}_2^{-1}\tilde{\phi}$ and $\psi = \mathfrak{L}_1^{-1}\tilde{\psi}$. We have $\phi, \psi \in \mathsf{Dom}\,(\mathfrak{B}_1) = \mathsf{Dom}\,(\mathfrak{B}_2)$. We notice that:

$$\mathfrak{B}_1(\phi,\psi) = \langle \mathfrak{L}_2^{-1}\tilde{\phi}, \tilde{\psi}\rangle, \quad \mathfrak{B}_2(\phi,\psi) = \langle \mathfrak{L}_1^{-1}\tilde{\phi}, \tilde{\psi}\rangle$$

and:

$$\mathfrak{Q}_1(\psi) = \langle \tilde{\psi}, \mathfrak{L}_1^{-1}\tilde{\psi}\rangle, \quad \mathfrak{Q}_2(\phi) = \langle \tilde{\phi}, \mathfrak{L}_2^{-1}\tilde{\phi}\rangle.$$

We infer that:

$$\left|\langle(\mathfrak{L}_1^{-1} - \mathfrak{L}_2^{-1})\tilde{\phi}, \tilde{\psi}\rangle\right| \leq \eta\|\mathfrak{L}_1^{-1}\|^{1/2}\|\mathfrak{L}_2^{-1}\|^{1/2}\|\tilde{\phi}\|\|\tilde{\psi}\|$$

and the result elementarily follows. □

1.7. A magnetic Hardy inequality. In dimension 2, the limiting model (with $\delta = 1$) highlights the fact that the magnetic field plays against the curvature, whereas in dimension 3 this repulsive effect is not obvious (it can be seen that $\langle D_\alpha R_\omega, J_1\rangle_\omega \geq 0$). Nevertheless, if ω is a disk, we have $\langle D_\alpha R_\omega, J_1\rangle_\omega = 0$ and thus the component of the magnetic field parallel to γ plays against the curvature (in comparison, a pure torsion has no effect when the cross section is a disk). In the flat case ($\kappa = 0$), we can quantify this repulsive effect by means of a magnetic Hardy inequality (see [40] where this inequality is discussed in dimension two).

Theorem 4.9. *Let $d \geq 2$ and ω be an open bounded subset of $\mathbb{R}^{d-1}$. Let us consider $\Omega = \mathbb{R} \times \omega$. For $R > 0$, we let:*

$$\Omega(R) = \{t \in \Omega : |t_1| < R\}.$$

Let $\boldsymbol{A}$ be a smooth vector potential such that the magnetic 2-form $\sigma_{\boldsymbol{B}}$ is not zero on $\Omega(R_0)$ for some $R_0 > 0$. Then, there exists $C > 0$ such that, for all $R \geq R_0$, there exists $c_R(\boldsymbol{B}) > 0$ such that, we have:

$$\int_\Omega |(-i\nabla + \boldsymbol{A})\psi|^2 - \lambda_1^{\mathsf{Dir}}(\omega)|\psi|^2\,\mathrm{d}t \geq \int_\Omega \frac{c_R(\boldsymbol{B})}{1+s^2}|\psi|^2\,\mathrm{d}t, \quad \forall\psi \in \mathcal{C}_0^\infty(\Omega). \tag{4.1.10}$$

Moreover we can take:

$$c_R(\boldsymbol{B}) = \left(1 + CR^{-2}\right)^{-1} \min\left(\frac{1}{4}, \lambda_1^{\mathsf{Dir,Neu}}(\boldsymbol{B}, \Omega(R)) - \lambda_1^{\mathsf{Dir}}(\omega)\right),$$

where $\lambda_1^{\mathsf{Dir,Neu}}(\boldsymbol{B}, \Omega(R))$ denotes the first eigenvalue of the magnetic Laplacian on $\Omega(R)$, with Dirichlet condition on $\mathbb{R} \times \partial\omega$ and Neumann condition on $\{|s| = R\} \times \omega$.

The inequality of Theorem 4.9 can be applied to prove certain stability of the spectrum of the magnetic Laplacian on Ω under local and small deformations of Ω. Let us fix $\varepsilon > 0$ and describe a generic deformation of the straight tube Ω. We consider the local diffeomorphism:

$$\Phi_\varepsilon(t) = \Phi_\varepsilon(s, t_2, t_3) = (s, 0, \cdots, 0) + \sum_{j=2}^{d} (t_j + \varepsilon_j(s)) M_j + \mathcal{E}_1(s),$$

where $(M_j)_{j=2}^{d}$ is the canonical basis of $\{0\} \times \mathbb{R}^{d-1}$. The functions ε_j and $\mathcal{E}_1$ are smooth and compactly supported in a compact set K. As previously we assume that Φ_ε is a global diffeomorphism and we consider the deformed tube $\Omega^{\mathsf{def},\varepsilon} = \Phi_\varepsilon(\mathbb{R} \times \omega)$.

Proposition 4.10. *Let $d \geq 2$. There exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$, the spectrum of the Dirichlet realization of $(-i\nabla + \boldsymbol{A})^2$ on $\Omega^{\mathsf{def},\varepsilon}$ coincides with the spectrum of the Dirichlet realization of $(-i\nabla + \boldsymbol{A})^2$ on Ω. The spectrum is given by $[\lambda_1^{\mathsf{Dir}}(\omega), +\infty)$.*

By using a semiclassical argument, it is possible to prove a stability result which does not use the Hardy inequality.

Proposition 4.11. *Let $R_0 > 0$ and $\Omega(R_0) = \{t \in \mathbb{R} \times \omega : |t_1| \leq R_0\}$. Let us assume that $\sigma_{\boldsymbol{B}} = \mathrm{d}\xi_{\boldsymbol{A}}$ does not vanish on $\Phi(\Omega(R_0))$ and that on $\Omega_1 \setminus \Phi(\Omega(R_0))$ the curvature is zero. Then, there exists $b_0 > 0$ such that for $b \geq b_0$, the discrete spectrum of $\mathfrak{L}_{1,b\boldsymbol{A}}^{[d]}$ is empty.*

2. Magnetic layers

As we will sketch below, the philosophy of Duclos and Exner may also apply to thin quantum layers as we can see in the contributions [**38**, **24**, **97**, **98**, **99**, **125**] and the related papers [**88**, **30**, **31**, **129**, **109**, **59**, **56**, **131**, **130**, **96**, **94**]. The collaboration with D. Krejčiřík and M. Tušek [**KRT13**] aimed at investigating the effect of a magnetic field.

Let us consider Σ an hypersurface embedded in $\mathbb{R}^d$ with $d \geq 2$, and define a tubular neighbourhood about Σ,

$$\Omega_\varepsilon := \left\{x + t\boldsymbol{n} \in \mathbb{R}^d \;\middle|\; (x, t) \in \Sigma \times (-\varepsilon, \varepsilon)\right\}, \tag{4.2.1}$$

where $\boldsymbol{n}$ denotes a unit normal vector field of Σ. We investigate:

$$\mathcal{L}_{\mathbf{A},\Omega_\varepsilon} = (-i\nabla + \mathbf{A})^2 \qquad \text{on} \qquad \mathsf{L}^2(\Omega_\varepsilon), \tag{4.2.2}$$

with Dirichlet boundary conditions on $\partial\Omega_\varepsilon$.

2.1. Normal form. As usual the game is to find an appropriate normal form for the magnetic Laplacian. Given $I := (-1,1)$ and $\varepsilon > 0$, we define a layer Ω_ε of width 2ε along Σ as the image of the mapping

$$\Phi : \Sigma \times I \to \mathbb{R}^d : \ \big\{(x,u) \mapsto x + \varepsilon u \boldsymbol{n}\big\} \tag{4.2.3}$$

Let us denote by $\tilde{\mathbf{A}}$ the components of the vector potential expressed in the curvilinear coordinates induced by the embedding (4.2.3). Moreover, assume

$$\tilde{A}_d = 0. \tag{4.2.4}$$

Thanks to the diffeomorphism $\Phi : \Sigma \times I \to \Omega_\varepsilon$, we may identify $\mathcal{L}_{\mathbf{A},\Omega_\varepsilon}$ with an operator $\hat{H}$ on $\mathsf{L}^2(\Sigma \times I, \, \mathrm{d}\Omega_\varepsilon)$ that acts, in the form sense, as

$$\hat{H} = |G|^{-1/2}(-i\partial_{x^\mu} + \tilde{A}_\mu)|G|^{1/2}G^{\mu\nu}(-i\partial_{x^\nu} + \tilde{A}_\nu) - \varepsilon^{-2}|G|^{-1/2}\partial_u|G|^{1/2}\partial_u \,.$$

Let us define

$$J := \frac{1}{4}\ln\frac{|G|}{|g|} = \frac{1}{2}\sum_{\mu=1}^{d-1}\ln(1-\varepsilon u\kappa_\mu) = \frac{1}{2}\ln\left[1 + \sum_{\mu=1}^{d-1}(-\varepsilon u)^\mu \binom{d-1}{\mu} K_\mu\right].$$

Using the unitary transform

$$U : \mathsf{L}^2(\Sigma \times I, \, \mathrm{d}\Omega_\varepsilon) \to \mathsf{L}^2(\Sigma \times I, \, \mathrm{d}\Sigma \wedge \, \mathrm{d}u) : \ \left\{\psi \mapsto e^J\psi\right\} \,,$$

we arrive at the unitarily equivalent operator

$$H := U\hat{H}U^{-1} = |g|^{-1/2}(-i\partial_{x^\mu} + \tilde{A}_\mu)|g|^{1/2}G^{\mu\nu}(-i\partial_{x^\nu} + \tilde{A}_\nu) - \varepsilon^{-2}\partial_u^2 + V \,,$$

where

$$V := |g|^{-1/2}\,\partial_{x^i}\big(|g|^{1/2}G^{ij}(\partial_{x^j}J)\big) + (\partial_{x^i}J)G^{ij}(\partial_{x^j}J) \,.$$

We get

$$H = U\hat{U}(-\Delta_{D,A}^{\Omega_\varepsilon})\hat{U}^{-1}U^{-1} \,.$$

2.2. The effective operator. H is approximated in the norm resolvent sense (see [**KRT13**] for the details) by

$$H_0 = h_{\text{eff}} - \varepsilon^{-2}\partial_u^2 \ \simeq \ h_{\text{eff}} \otimes 1 + 1 \otimes (-\varepsilon^{-2}\partial_u^2) \tag{4.2.5}$$

on $\mathsf{L}^2(\Sigma \times I, \, \mathrm{d}\Sigma \wedge \, \mathrm{d}u) \simeq \mathsf{L}^2(\Sigma, \, \mathrm{d}\Sigma) \otimes \mathsf{L}^2(I, \, \mathrm{d}u)$ with the effective Hamiltonian

$$h_{\text{eff}} := |g|^{-1/2}\big(-i\partial_{x^\mu} + \tilde{A}_\mu(.,0)\big)|g|^{1/2}g^{\mu\nu}\big(-i\partial_{x^\nu} + \tilde{A}_\nu(.,0)\big) + V_{\text{eff}}, \tag{4.2.6}$$

where

$$V_{\text{eff}} := -\frac{1}{2}\sum_{\mu=1}^{d-1}\kappa_\mu^2 + \frac{1}{4}\left(\sum_{\mu=1}^{d-1}\kappa_\mu\right)^2 . \tag{4.2.7}$$

In particular the effective Hamiltonian only feels the normal component of the magnetic field.

3. Semiclassical triangles

As we would like to analyze the spectrum of broken waveguides (that is waveguides with an angle), this is natural to prepare the investigation by studying the Dirichlet eigenvalues of the Laplacian on some special shrinking triangles. This subject is already dealt with in [55, Theorem 1] where four-term asymptotics is proved for the lowest eigenvalue, whereas a three-term asymptotics for the second eigenvalue is provided in [55, Section 2]. We can mention the papers [57, 58] whose results provide two-term asymptotics for the thin rhombi and also [16] which deals with a regular case (thin ellipse for instance), see also [17]. We also invite the reader to take a look at [84]. For a complete description of the low lying spectrum of general shrinking triangles, one may consult the paper by my student Ourmières [117] where tunnel effect estimates are also established. In dimension three the generalization to cones with small aperture is done in [116] and is motivated by [49]. The result of this section was obtained in the collaboration with M. Dauge [DaR12] and aimed at applying the semiclassical techniques to this kind of geometric problems. In particular, this work establishes that, in the case of shrinking triangles, the eigenfunctions contain a boundary layer (they live on different scales depending on h). This fact does not seem to be known in the literature.

Let us define the isosceles triangle in which we are interested:

$$\mathsf{Tri}_\theta = \left\{(x_1, x_2) \in \mathbb{R}_- \times \mathbb{R} : x_1 \tan\theta < |x_2| < \left(x_1 + \frac{\pi}{\sin\theta}\right)\tan\theta\right\}. \tag{4.3.1}$$

We will use the coordinates

$$x = x_1\sqrt{2}\sin\theta, \quad y = x_2\sqrt{2}\cos\theta, \tag{4.3.2}$$

which transform Tri_θ into $\mathsf{Tri}_{\pi/4}$. The operator becomes:

$$\mathcal{D}_{\mathsf{Tri}}(h) = 2\sin^2\theta\,\partial_x^2 - 2\cos^2\theta\,\partial_y^2,$$

with Dirichlet condition on the boundary of Tri. We let $h = \tan\theta$; after a division by $2\cos^2\theta$, we get the new operator:

$$\mathcal{L}_{\mathsf{Tri}}(h) = -h^2\partial_x^2 - \partial_y^2. \tag{4.3.3}$$

This operator is thus in the "Born-Oppenheimer form" and we shall introduce its Born-Oppenheimer approximation which is the Dirichlet realization on $\mathsf{L}^2((-\pi\sqrt{2}, 0))$ of:

$$\mathcal{H}_{\mathsf{BO},\mathsf{Tri}}(h) = -h^2\partial_x^2 + \frac{\pi^2}{4(x+\pi\sqrt{2})^2}. \tag{4.3.4}$$

Theorem 4.12. *The eigenvalues of $\mathcal{H}_{\mathsf{BO},\mathsf{Tri}}(h)$, denoted by $\lambda_{\mathsf{BO},\mathsf{Tri},n}(h)$, admit the expansions:*

$$\lambda_{\mathsf{BO},\mathsf{Tri},n}(h) \underset{h\to 0}{\sim} \sum_{j\geq 0} \hat{\beta}_{j,n} h^{2j/3}, \quad \textit{with} \quad \hat{\beta}_{0,n} = \frac{1}{8} \quad \textit{and} \quad \hat{\beta}_{1,n} = (4\pi\sqrt{2})^{-2/3} z_{\mathsf{Ai}^{\mathsf{rev}}}(n),$$

where $z_{\mathsf{Ai}^{\mathsf{rev}}}(n)$ is the n-th zero of the reversed Airy function $\mathsf{Ai}^{\mathsf{rev}}(x) = \mathsf{Ai}(-x)$.

We state the result for the scaled operator $\mathcal{L}_{\mathsf{Tri}}(h)$.

Theorem 4.13. *The eigenvalues of $\mathcal{L}_{\mathsf{Tri}}(h)$, denoted by $\lambda_{\mathsf{Tri},n}(h)$, admit the expansions:*

$$\lambda_{\mathsf{Tri},n}(h) \underset{h\to 0}{\sim} \sum_{j\geq 0} \beta_{j,n} h^{j/3} \quad \textit{with} \quad \beta_{0,n} = \frac{1}{8}, \quad \beta_{1,n} = 0, \quad \textit{and} \quad \beta_{2,n} = (4\pi\sqrt{2})^{-2/3} z_{\mathsf{Ai}^{\mathsf{rev}}}(n),$$

the terms of odd rank being zero for $j \leq 8$. The corresponding eigenfunctions have expansions in powers of $h^{1/3}$ with both scales $x/h^{2/3}$ and x/h (the "boundary layer").

4. Broken waveguides

4.1. Physical motivation. As we have already recalled at the beginning of this chapter, it has been proved in [37] that a curved, smooth and asymptotically straight waveguide has discrete spectrum below its essential spectrum. Now we would like to explain the influence of a corner which is somehow an infinite curvature and extend the philosophy of the smooth case. This is the aim of the collaboration with M. Dauge [DaR12] (see also the numerical counterpart in [DaLR11]), especially by applying the semiclassical methods in the context of waveguides.

This question is investigated with the L-shape waveguide in [48] where the existence of discrete spectrum is proved. For an arbitrary angle too, this existence is proved in [4] and an asymptotic study of the ground energy is done when θ goes to $\frac{\pi}{2}$ (where θ is the semi-opening of the waveguide). Another question which arises is the estimate of the lowest eigenvalues in the regime $\theta \to 0$. This problem is analyzed in [23] where a waveguide with corner is the model chosen to describe some electromagnetic experiments (see Figure 2).

4.2. Geometric description. Let us denote by (x_1, x_2) the Cartesian coordinates of the plane and by $\mathbf{0} = (0,0)$ the origin. Let us define our so-called "broken waveguides". For any angle $\theta \in \left(0, \frac{\pi}{2}\right)$ we introduce

$$\Omega_\theta = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \tan\theta < |x_2| < \left(x_1 + \frac{\pi}{\sin\theta}\right)\tan\theta\right\}. \tag{4.4.1}$$

Note that its width is independent from θ, normalized to π, see Figure 3. The limit case where $\theta = \frac{\pi}{2}$ corresponds to the straight strip $(-\pi, 0) \times \mathbb{R}$.

The operator $-\Delta^{\mathsf{Dir}}_{\Omega_\theta}$ is a positive unbounded self-adjoint operator with domain

$$\mathsf{Dom}\,(-\Delta^{\mathsf{Dir}}_{\Omega_\theta}) = \{\psi \in \mathsf{H}^1_0(\Omega_\theta) : \quad -\Delta\psi \in \mathsf{L}^2(\Omega_\theta)\}.$$

When $\theta \in \left(0, \frac{\pi}{2}\right)$, the boundary of Ω_θ is not smooth, it is polygonal. The presence of the non-convex corner with vertex $\mathbf{0}$ is the reason for the space $\mathsf{Dom}\,(-\Delta^{\mathsf{Dir}}_{\Omega_\theta})$ to be distinct from $\mathsf{H}^2 \cap \mathsf{H}^1_0(\Omega_\theta)$. We have the following description of the domain (see the classical references [91, 61]):

$$\mathsf{Dom}\,(-\Delta^{\mathsf{Dir}}_{\Omega_\theta}) = \left(\mathsf{H}^2 \cap \mathsf{H}^1_0(\Omega_\theta)\right) \oplus [\psi^\theta_{\mathsf{sing}}] \tag{4.4.2}$$

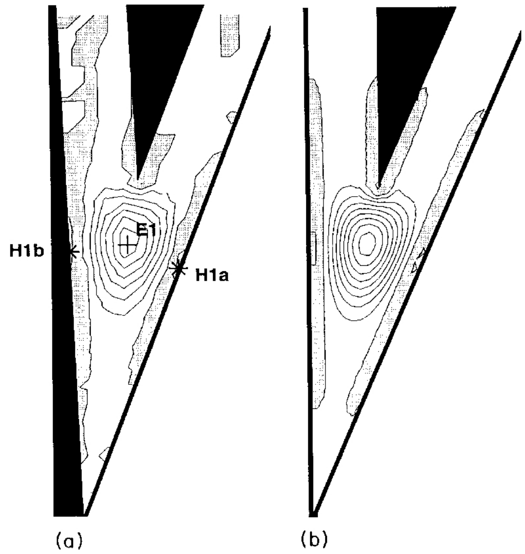


FIG. 6. (a) Experimentally measured contour plots of frequency shifts for lowest-frequency bound-state wave functions for a sharply bent waveguide with an interior angle $\theta = 22.5°$. The frequency shift measured as a function of position for a small metal sphere inside the waveguide. Shaded areas denote regions of positive frequency shift (relatively large magnetic field energy density); unshaded areas signify negligible or negative frequency shift (negligible or relatively large electric-field energy density). Point $E1$ denotes the maximum negative-energy shift (antinode of E_z); points $H1a$ and $H1b$ denote points of the maximum positive-energy shift (antinodes of H_t). Numerical values of these quantities are given in Table II. (b) Calculations of the same quantity shown in (a), using Eq. (23).

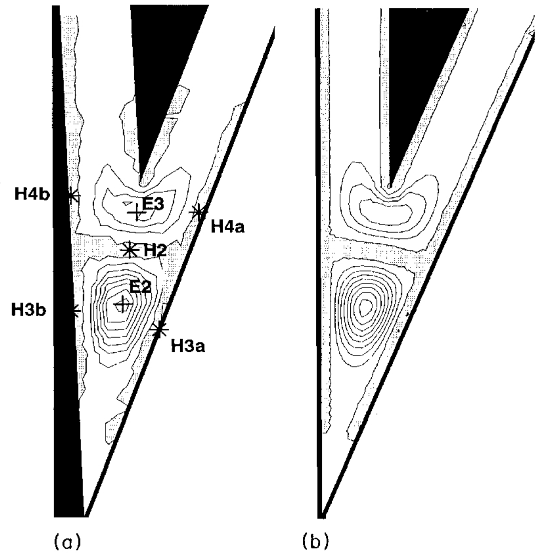


FIG. 7. Experimentally measured contour plots for the frequency shift for the first excited state of a bent waveguide with an interior angle $\theta = 22.5°$. The notation is that of Fig. 6. Points $E2$ and $E3$ denote maximum measured negative frequency shifts; points $H2$, $H3a$, $H3b$, $H4a$, and $H4b$ represent local maxima in the frequency shifts. (b) Calculations of the frequency shift for the same quantity shown in (a).

FIGURE 2. Experimental results of [23]

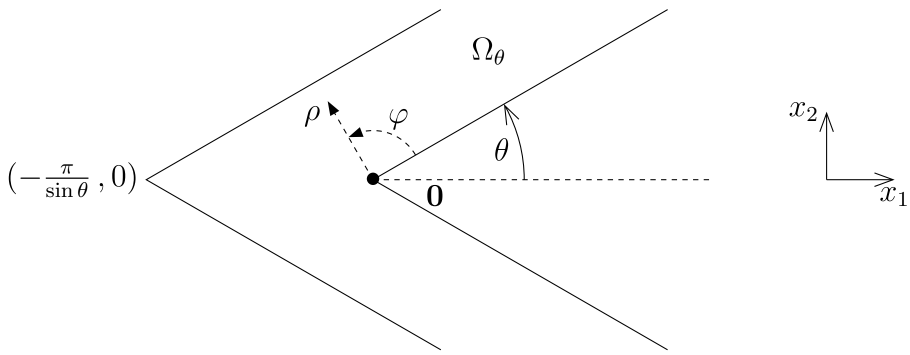


FIGURE 3. The broken guide Ω_θ (here $\theta = \frac{\pi}{6}$). Cartesian and polar coordinates.

where $[\psi^\theta_{\mathsf{sing}}]$ denotes the space generated by the singular function $\psi^\theta_{\mathsf{sing}}$ defined in the polar coordinates (ρ, φ) near the origin by

$$\psi^\theta_{\mathsf{sing}}(x_1, x_2) = \chi(\rho)\, \rho^{\pi/\omega} \sin\frac{\pi\varphi}{\omega} \quad \text{with} \quad \omega = 2(\pi - \theta) \tag{4.4.3}$$

where where χ is a radial cutoff function near the origin.

We gather in the following statement several important preliminary properties for the spectrum of $-\Delta^{\mathsf{Dir}}_{\Omega_\theta}$. All these results are proved in the literature (see also [DaLR11]).

Proposition 4.14. (i) *If $\theta = \frac{\pi}{2}$, $-\Delta^{\mathsf{Dir}}_{\Omega_\theta}$ has no discrete spectrum. Its essential spectrum is the closed interval $[1, +\infty)$.*

(ii) *For any θ in the open interval $(0, \frac{\pi}{2})$ the essential spectrum of $-\Delta^{\mathsf{Dir}}_{\Omega_\theta}$ coincides with $[1, +\infty)$.*

(iii) *For any $\theta \in (0, \frac{\pi}{2})$, the discrete spectrum of $-\Delta^{\mathsf{Dir}}_{\Omega_\theta}$ is* nonempty*.*

(iv) *For any $\theta \in (0, \frac{\pi}{2})$ and any eigenvalue in the discrete spectrum of $-\Delta^{\mathsf{Dir}}_{\Omega_\theta}$, the associated eigenvectors ψ are* even *with respect to the horizontal axis: $\psi(x_1, -x_2) = \psi(x_1, x_2)$.*

(v) *For any $\theta \in (0, \frac{\pi}{2})$, let $\mu_{\mathsf{Gui},n}(\theta)$, $n = 1, \ldots,$ be the n-th Rayleigh quotient of $-\Delta^{\mathsf{Dir}}_{\Omega_\theta}$. Then, for any $n \geq 1$, the function $\theta \mapsto \mu_{\mathsf{Gui},n}(\theta)$ is continuous and increasing.*

It is also possible to prove that the number of eigenvalues below the essential spectrum is exactly 1 as soon as θ is close enough to $\frac{\pi}{2}$ (see [**115**]). In [**DaLR11**], we provide a proof of the following proposition (which is inspired by [**112**, Theorem 2.1]).

Proposition 4.15. *For any $\theta \in (0, \frac{\pi}{2})$, the number of eigenvalues of $-\Delta^{\mathsf{Dir}}_{\Omega_\theta}$ below 1, denoted by $\mathsf{N}(-\Delta^{\mathsf{Dir}}_{\Omega_\theta}, 1)$, is finite.*

As a consequence of the parity properties of the eigenfunctions of $-\Delta^{\mathsf{Dir}}_{\Omega_\theta}$, cf. point (iv) of Proposition 4.14, we can reduce the spectral problem to the half-guide

$$\Omega^+_\theta = \{(x_1, x_2) \in \Omega_\theta : \ x_2 > 0\}. \tag{4.4.4}$$

We define the Dirichlet part of the boundary by $\partial_{\mathsf{Dir}}\Omega^+_\theta = \partial\Omega_\theta \cap \partial\Omega^+_\theta$, and the form domain

$$\mathsf{H}^1_{\mathsf{Mix}}(\Omega^+_\theta) = \big\{\psi \in \mathsf{H}^1(\Omega^+_\theta) : \quad \psi = 0 \ \text{ on } \ \partial_{\mathsf{Dir}}\Omega^+_\theta\big\}.$$

Then the new operator of interest, denoted by $-\Delta^{\mathsf{Mix}}_{\Omega^+_\theta}$, is the Laplacian with mixed Dirichlet-Neumann conditions on Ω^+_θ. Its domain is:

$$\mathsf{Dom}\,(-\Delta^{\mathsf{Mix}}_{\Omega^+_\theta}) = \big\{\psi \in \mathsf{H}^1_{\mathsf{Mix}}(\Omega^+_\theta) : \quad \Delta\psi \in \mathsf{L}^2(\Omega^+_\theta) \ \text{ and } \ \partial_2\psi = 0 \ \text{ on } \ x_2 = 0\big\}.$$

Then the operators $-\Delta^{\mathsf{Dir}}_{\Omega_\theta}$ and $-\Delta^{\mathsf{Mix}}_{\Omega^+_\theta}$ have the same eigenvalues below 1 and the eigenfunctions of the latter are the restriction to Ω^+_θ of the former.

In order to analyze the asymptotics $\theta \to 0$, it is useful to rescale the integration domain and transfer the dependence on θ into the coefficients of the operator. For this reason, let us perform the following linear change of coordinates:

$$x = x_1\sqrt{2}\sin\theta, \quad y = x_2\sqrt{2}\cos\theta, \tag{4.4.5}$$

which maps Ω^+_θ onto the θ-independent domain $\Omega^+_{\pi/4}$, see Fig. 4. That is why we set for simplicity

$$\Omega := \Omega^+_{\pi/4}, \quad \partial_{\mathsf{Dir}}\Omega = \partial_{\mathsf{Dir}}\Omega^+_{\pi/4}, \quad \text{and} \quad \mathsf{H}^1_{\mathsf{Mix}}(\Omega) = \big\{\psi \in \mathsf{H}^1(\Omega) : \psi = 0 \ \text{ on } \ \partial_{\mathsf{Dir}}\Omega\big\}. \tag{4.4.6}$$

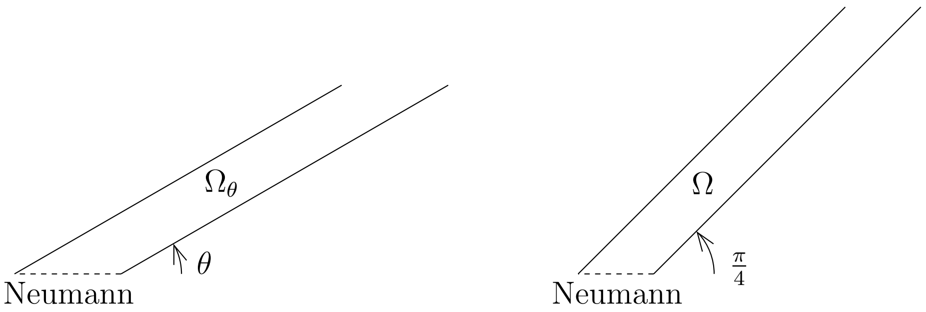


FIGURE 4. The half-guide Ω_θ^+ for $\theta = \frac{\pi}{6}$ and the reference domain Ω.

Then, $\Delta^{\mathsf{Mix}}_{\Omega_\theta^+}$ is unitarily equivalent to the operator defined on Ω by:

$$\mathcal{D}_{\mathsf{Gui}}(\theta) := -2\sin^2\theta\,\partial_x^2 - 2\cos^2\theta\,\partial_y^2, \tag{4.4.7}$$

with Neumann condition on $y = 0$ and Dirichlet everywhere else on the boundary of Ω. We let $h = \tan\theta$; after a division by $2\cos^2\theta$, we get the new operator:

$$\mathcal{L}_{\mathsf{Gui}}(h) = -h^2\partial_x^2 - \partial_y^2, \tag{4.4.8}$$

with domain:

$$\mathsf{Dom}\,(\mathcal{L}_{\mathsf{Gui}}(h)) = \left\{\psi \in \mathsf{H}^1_{\mathsf{Mix}}(\Omega) : \quad \mathcal{L}_{\mathsf{Gui}}(h)\psi \in L^2(\Omega) \ \text{ and } \ \partial_y\psi = 0 \ \text{ on } \ y = 0\right\}.$$

The Born-Oppenheimer approximation is:

$$\mathcal{H}_{\mathsf{BO,Gui}}(h) = -h^2\partial_x^2 + V(x), \tag{4.4.9}$$

where

$$V(x) = \begin{cases} \dfrac{\pi^2}{4(x+\pi\sqrt{2})^2} & \text{when } x \in (-\pi\sqrt{2}, 0), \\ \dfrac{1}{2} & \text{when } x \geq 0. \end{cases}$$

4.3. Eigenvalues induced by a strongly broken waveguide. Let us now state the main result concerning the asymptotic expansion of the eigenvalues of the broken waveguide.

Theorem 4.16. *For all N_0, there exists $h_0 > 0$, such that for $h \in (0, h_0)$ the N_0 first eigenvalues of $\mathcal{L}_{\mathsf{Gui}}(h)$ exist. These eigenvalues, denoted by $\lambda_{\mathsf{Gui},n}(h)$, admit the expansions:*

$$\lambda_{\mathsf{Gui},n}(h) \underset{h\to 0}{\sim} \sum_{j\geq 0} \gamma_{j,n} h^{j/3} \quad \text{with} \ \ \gamma_{0,n} = \frac{1}{8}, \quad \gamma_{1,n} = 0, \quad \text{and} \ \ \gamma_{2,n} = (4\pi\sqrt{2})^{-2/3} z_{\mathsf{A}\mathsf{i}^{\mathsf{rev}}}(n)$$

and the term of order h is not zero. The corresponding eigenvectors have expansions in powers of $h^{1/3}$ with the scale x/h when $x > 0$, and both scales $x/h^{2/3}$ and x/h when $x < 0$.

The main ingredient in the proof of Theorem 4.16 is the construction of quasimodes. This one uses double scale expansions and the Dirichlet-to-Neumann operators to handle the connection between the triangle part and the guiding part of the waveguide.

4.4. A few numerical simulations from [DaR12]. Let us provide some numerical simulations (using Melina [102]) of the first eigenfunctions.

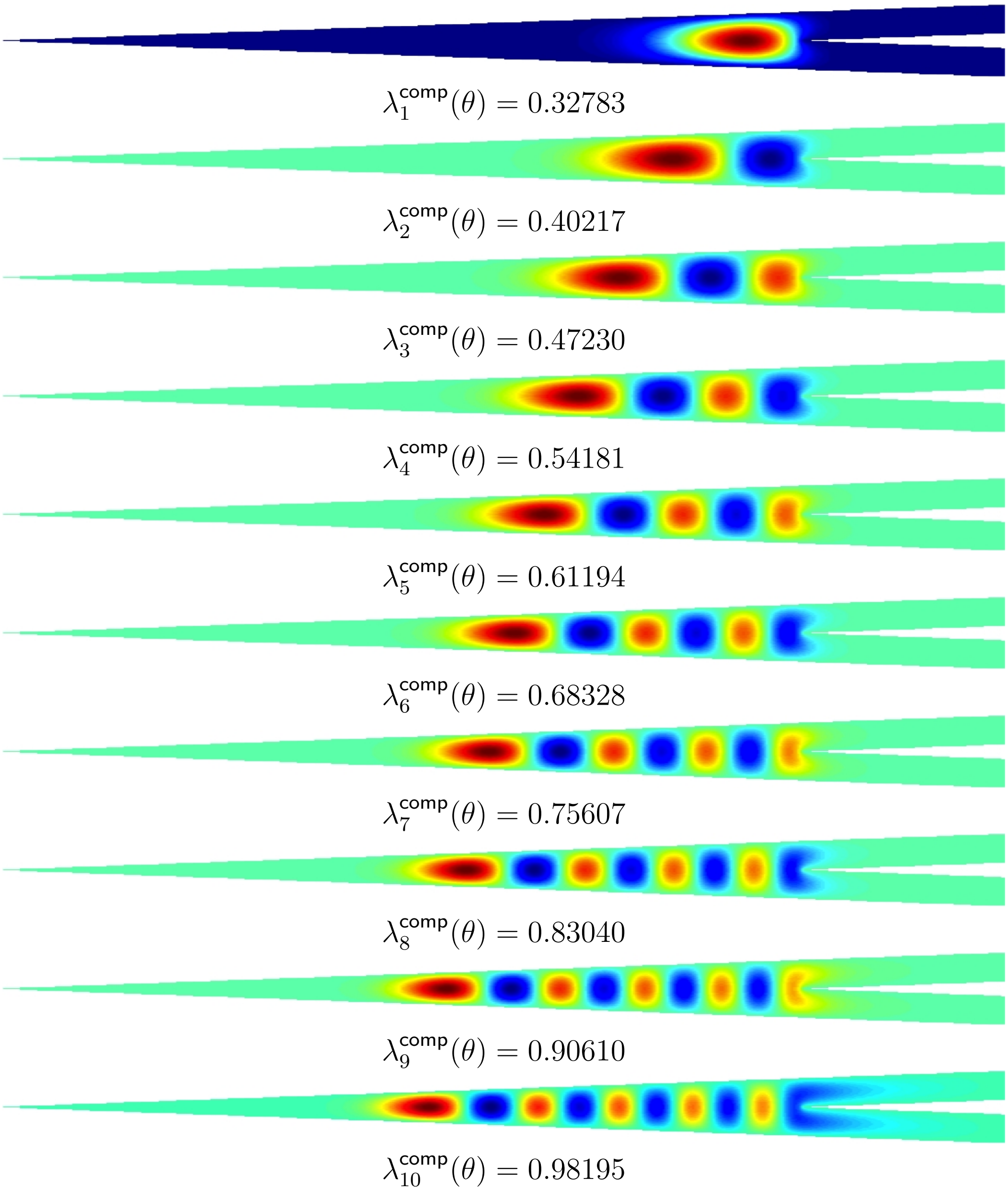


FIGURE 5. Computations for $\theta = 0.0226 * \pi/2 \sim 2^\circ$ with the mesh M64S. Numerical values of the 10 eigenvalues $\lambda_j(\theta) < 1$. Plots of the associated eigenfunctions in the physical domain.

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